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THE IMPERFECTLY CONDUCTING COAXIAL LINE*

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Abstract. In order to determine the range of validity of certain elementary concepts in waveguide theory, the propagation of electromagnetic waves along and through a coaxial line with imperfectly conducting walls is studied in some detail for a particular method of driving. In particular, it is found that the usual concept of attenuation is meaningful only for a certain range of distances from the driving point. Beyond this distance, the electromagnetic field in the coaxial line behaves more like a radiation field. The explanation is supported by the behavior of the electromagnetic field in the imperfect outer conductor of the coaxial line. It is also found that the solution in terms of the "mode" concept has a surprisingly limited region of validity. The reflection coefficient, the radiation pattern and the transverse distribution are also determined.

1. Introduction. The propagation of a wave through an imperfectly conducting coaxial line is of interest from the point of view of waveguide theory. In the elementary theory of waveguides, the wall of the waveguide is assumed to be perfectly conducting. Under this assumption, there arises the extremely important, useful, and convenient concept of "mode." With this concept, these waveguide problems become essentially two-dimensional, since for each mode the dependence on z (the direction of the waveguide) is simply exponential. Furthermore, the number of modes is countably infinite. However, the situation is no longer so simple if it is possible for the electromagnetic wave to reach infinity in a direction perpendicular to the z-axis, as it can when the waveguide is open or when the waveguide has an imperfectly conducting wall. In the latter case, it is sometimes possible to use the idea of an impedance wall to save the situation. Otherwise, the modes lose at least some of their desirable properties, the excellent work of Marcuvitz and coworkers [1] in this connection notwithstanding. As an example of the open waveguide, consider the microstrip. Here the lowest mode, although exponentially decreasing in the direction of the z-axis, is unfortunately exponentially increasing at infinity in any plane z = constant. This is readily understandable and is intimately connected with the infinite length of the line assumed in the theoretical treatment [2]. On the other hand, for any physical waveguide or transmission line, the total length must be finite and consequently any meaningful solution of the physical problem must satisfy the usual radiation condition of Sommerfeld. For

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this reason, the connection between the theoretical and the experimental aspects of the microstrip problem is by no means evident. Furthermore, it is apparently desirable to treat a problem of this type where the generator is not at infinity. No really simple problem of this type seems to have been treated in electromagnetic theory, and the only reasonable candidates seem to be the circular waveguide and the coaxial line. The advantage of the first is that it has one less dimension (the size of the inner conductor), while loosely the advantage of the second is that there is no mixing of the TE and the TM modes. Mathematically, the second problem is the simpler one of the two, and hence the imperfectly conducting coaxial line is treated in this paper.

The precise geometry of the problem is shown in Fig. 1, where the imperfectly conducting outer conductor of the coaxial line is of finite thickness and runs from $-\infty$ to $+\infty$ along the z-axis while immediately inside a perfect conductor of zero thickness is fitted from $-\infty$ to 0. To avoid meaningless complications, the inner conductor is assumed to be perfectly conducting from $-\infty$ to $+\infty$. It is assumed that a TEM wave propagates from left to right; it is incident on the junction at $x = 0$. Thus there is effectively a generator at $z = 0$. This geometry is chosen because, in principle, at least, it possesses a closed solution in terms of quadratures.

Since this problem is treated here only as an example of waveguide theory, there is no necessity of dealing with it in its full generality. Therefore, in the approximate theory of Secs. 5-10, it will be assumed that the free space wavelength is much larger than the transverse dimensions of the coaxial line in addition to the more specialized assumptions (5.1) and (5.2).

This problem is also of considerable interest in connection with the so-called electromagnetic shielding problem. However, in this connection, the assumptions (5.1) and (5.2) are too restrictive, and consequently, a more elaborate approximate theory has to be developed after a thorough understanding of the features of the more restrictive problem treated here has been acquired.

Before studying the problem of the imperfectly conducting coaxial line, it is necessary to know some of the properties of the Green's function to be used in the integral-equation formulation. This Green's function is studied in the next section.

2. Green's function. The interest here is in the implications associated with the finite conductivity of the outer conductor. Since these implications do not depend critically on the choice of the dielectric constant of this imperfect conductor, this dielectric constant is to be identified with that of vacuum in the following consideration. If the conductor is also non-magnetic, then both the magnetic permeability and the dielectric

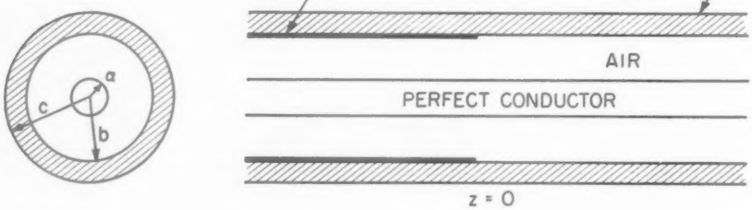

FIG. 1. Geometry of the problem.

constant take those values appropriate for vacuum, say μ and ϵ. With the time dependence $\exp(-i\omega t)$, Maxwell's equations are

$$
\begin{aligned}
\nabla \times \mathbf{E} &= i\omega\mu\mathbf{H}, \\
\nabla \times \mathbf{H} &= (\sigma - i\omega\epsilon)\mathbf{E} + \mathbf{J}, \\
\nabla \cdot \mathbf{E} &= \nabla\cdot\mathbf{H} = 0,
\end{aligned}
\tag{2.1}
$$

where σ is the conductivity of the imperfect outer conductor of the coaxial line and is replaced by 0 for air; the other symbols all have their usual meanings. If conventional cylindrical coordinates are used, all field quantities are independent of θ in this axially symmetric problem. Therefore

$$E_\theta = H_z = H_r = 0. \tag{2.2}$$

With the continuity of E_z and H_θ as boundary conditions and the radiation condition at infinity, the electromagnetic field is completely determined by (2.1) provided that $\mathbf{J}$ is given. Let $G_0(r, \zeta)\exp(i\zeta z)$ be the z-component of the electric field produced by the current source $\mathbf{J} = \mathbf{z}\,\delta(r - b)\exp(i\zeta z)$, where $\mathbf{z}$ is the unit vector in the z-direction; it may then be verified that

$$G_0(b, \zeta) = i\omega\mu[k^2\xi^{-1}Q_1 - (k^2 + i\kappa^2)\eta^{-1}Q_2]^{-1}, \tag{2.3}$$

where

$$Q_1 = \frac{H_0^{(2)}(\xi a)H_0^{(1)\prime}(\xi b) - H_0^{(1)}(\xi a)H_0^{(2)\prime}(\xi b)}{H_0^{(2)}(\xi a)H_0^{(1)}(\xi b) - H_0^{(1)}(\xi a)H_0^{(2)}(\xi b)}, \tag{2.4}$$

$$
\begin{aligned}
Q_2 = {} & \begin{vmatrix} H_c^{(1)}(\xi c) & H_0^{(1)}(\eta b)H_0^{(2)}(\eta c) - H_0^{(2)}(\eta b)H_0^{(1)}(\eta c) \\ \eta\xi^{-1}k^2(k^2 + i\kappa^2)^{-1}H_0^{(1)\prime}(\xi c) & H_0^{(1)}(\eta b)H_0^{(2)\prime}(\eta c) - H_0^{(2)}(\eta b)H_0^{(1)\prime}(\eta c) \end{vmatrix}^{-1} \\
& \times \begin{vmatrix} H_0^{(1)}(\xi c) & H_0^{(1)\prime}(\eta b)H_0^{(2)}(\eta c) - H_0^{(2)\prime}(\eta b)H_0^{(1)}(\eta c) \\ \eta\xi^{-1}k^2(k^2 + i\kappa^2)^{-1}H_0^{(1)\prime}(\xi c) & H_0^{(1)\prime}(\eta b)H_0^{(2)\prime}(\eta c) - H_0^{(2)\prime}(\eta b)H_0^{(1)\prime}(\eta c) \end{vmatrix},
\end{aligned}
\tag{2.5}
$$

$$\xi = (k^2 - \zeta^2)^{1/2}, \qquad \eta = (k^2 + i\kappa^2 - \zeta^2)^{1/2}, \tag{2.6}$$

and

$$k^2 = \omega^2\mu\epsilon, \qquad \kappa^2 = \omega\mu\sigma. \tag{2.7}$$

Note that

$$G_0(b, \zeta) = G_0(b, -\zeta), \tag{2.8}$$

and that

$$G_0(b, \zeta) \sim -i(2k^2 + i\kappa^2)^{-1}\omega\mu\zeta \tag{2.9}$$

as $\zeta \to \infty$. Because of (2.9), it is not meaningful to take the Fourier transform of $G_0(b, \zeta)$ with respect to ζ. Since $G_0(b, \zeta) = O(k^2 - \zeta^2)$ as $\zeta \to \pm k$, it is possible to introduce an analog of the ordinary vector potential by defining

$$g_1(b, z) = (2\pi)^{-1}\int_{-\infty}^{\infty} d\zeta G_1(b, \zeta)\exp(i\zeta z), \tag{2.10}$$

where

$$G_1(b, \zeta) = (k^2 - \zeta^2)^{-1}G_0(b, \zeta). \tag{2.11}$$

3. Formulation of the problem. The problem of the coaxial line may now be formulated in terms of a Wiener-Hopf integral equation. Let $i(z)$ be the z-component of the total surface current density on the infinitely thin perfectly conducting sleeve; the incident current density is $\exp(ikz)$. Thus as $z \to -\infty$

$$i(z) \sim \exp(ikz) + \Gamma \exp(-ikz), \tag{3.1}$$

where Γ is the coefficient of reflection with reference to the junction $z = 0$. With the g_1 of (2.10), $i(z)$ satisfies the integral equation

$$\left(\frac{d^2}{dz^2} + k^2\right) \int_{-\infty}^{0} dz' i(z') g_1(b, z - z') = 0 \tag{3.2}$$

for $z < 0$. If k is assumed to have a small positive imaginary part which eventually approaches zero, then the Fourier transform of (3.2) in the form

$$\int_{-\infty}^{0} dz'\left[\left(\frac{\partial}{\partial z'} - ik\right)i(z')\right]\left[\left(\frac{\partial}{\partial z} + ik\right)g_1(b, z - z')\right] = E_z(b, z) \tag{3.3}$$

is

$$-[(\zeta - k)I(\zeta)][(\zeta + k)G_1(b, \zeta)] = \mathcal{E}_z(b, \zeta). \tag{3.4}$$

This equation is valid in a small strip $|\operatorname{Im} \zeta| < \epsilon$, where $G_1(b, \zeta)$ has no zero. The standard Wiener-Hopf procedure calls for a factorization of G_1 in the form

$$G_1(b, \zeta) = L_+(\zeta)/L_-(\zeta), \tag{3.5}$$

where

$$L_\pm(\zeta) = \exp\left\{(2\pi i)^{-1} \int_{-\infty \mp i\epsilon/2}^{\infty \mp i\epsilon/2} dt(t - \zeta)^{-1} \ln G_1(b, t)\right\}. \tag{3.6}$$

Note that

$$L_-(\zeta) = [L_+(-\zeta)]^{-1}. \tag{3.7}$$

It follows from (3.4) and (3.5) that

$$(k^2 - \zeta^2)I(\zeta)L_+(\zeta) = \mathcal{E}_z(b, \zeta)L_-(\zeta). \tag{3.8}$$

This defines an integral function which is found to be a constant, say C. As a consequence of (3.1), the special case $\zeta = k$ gives

$$C = 2ikL_+(k). \tag{3.9}$$

From (3.8) the electric field is given by

$$E_z(b, z) = (2\pi)^{-1}C \int_{-\infty}^{\infty} d\zeta[L_-(\zeta)]^{-1} \exp(i\zeta z). \tag{3.10}$$

In principle, the problem has been solved in closed form.

4. The radiation pattern. The field far away from the origin is considered here. Since as $\xi \to 0$

$$G_0(b, \zeta) = -(i\omega\epsilon)^{-1}[b \ln(b/2)]\xi^2[1 + O(\xi^4 \ln \xi)], \tag{4.1}$$

$H_\theta(b, z)$ must be proportional to z^{-2} as $z \to \infty$. This fact is of interest because it shows immediately that, as $z \to \infty$, an infinite number of "modes" must contribute significantly. The reason is that the combination of a finite number of modes must yield an exponential decay in the z-direction. This z^{-2} decay may be understood as follows. Because the outer conductor of the coaxial line is not perfectly conducting, there is a coupling of the field inside the coaxial line with the radiation field outside of the coaxial line. Since the outside field decays as z^{-2} because of cylindrical symmetry, the field inside the coaxial line must also decay as z^{-2}.

Thus the term "radiation pattern" is meaningful for the present problem. It follows from (3.8) and (3.9) that, as $r, z \to \infty$ with fixed $\phi = \tan^{-1}(r/z)$,

$$H_\theta(r, z) \sim A(\phi)(r^2 + z^2)^{-1/2} \exp [ik(r^2 + z^2)^{1/2}], \tag{4.2}$$

where

$$\begin{aligned} A(\phi) = \frac{8i\omega\epsilon}{\pi^2 c\Phi} \frac{L_+(k)}{L_-(k \cos \phi)} \{&H_0^{(1)}(kc \sin \phi) \sin \phi[H_0^{(1)}(b\Phi)H_0^{(2)\prime}(c\Phi) - H_0^{(2)}(b\Phi)H_0^{(1)\prime}(c\Phi)] \\ &- k(k^2 + i\kappa^2)^{-1}\Phi H_0^{(1)\prime}(kc \sin \phi)[H_0^{(1)}(b\Phi)H_0^{(2)}(c\Phi) - H_0^{(2)}(b\Phi)H_0^{(1)}(c\Phi)]\}^{-1}, \end{aligned} \tag{4.3}$$

with

$$\Phi = (k^2 \sin^2 \phi + i\kappa^2)^{1/2}. \tag{4.4}$$

5. Approximations for small skin depth. Although the exact solution has been obtained in principle, it is not very useful because of its complexity. The situation is greatly simplified if the coaxial line is only slightly leaky as when $c - b$ is much larger than skin depth $\delta = \kappa^{-1}\sqrt{2}$, or more precisely, if

$$b \gg \delta \tag{5.1}$$

and

$$\exp [\kappa(c - b) \sqrt{2}] \gg 1. \tag{5.2}$$

Under these assumptions, Q_2 of (2.5) is approximately

$$Q_2^{(1)} = i. \tag{5.3}$$

Furthermore when $kc \ll 1$, (5.3) leads to the following approximation of $G_0(b, \zeta)$

$$G_0^{(1)}(b, \zeta) = i\eta\sigma^{-1}[1 + C_1(k/\xi)^2]^{-1}, \tag{5.4}$$

where

$$C_1 = e^{i\pi/4}[kb \ln (b/a)]^{-1}. \tag{5.5}$$

The physical meaning of (5.2) is as follows. When the coaxial line is lossless, the characteristic scale of length for the variation of the field vectors in the z-direction is k^{-1}, which is very large. When the coaxial line is slightly leaky, the field inside the line tends to decrease when z increases, and this characteristic scale of length is the smaller one of k^{-1} and $\kappa bk^{-1} \exp [\kappa(c - b) \sqrt{2}]$. If this characteristic length is much larger than the outer radius c, then the transverse problem is essentially separated from the longitudinal problem, except possibly near $z = 0$. Once the longitudinal and the transverse problems are separated, the Green's function G_0 must admit simplification, and basically only the longitudinal problem need be considered. The condition (5.2) implies that the

frequency under consideration should not be too low in order that the thickness of the imperfect outer conductor be several times the skin depth.

When k^2 is neglected compared with κ^2, (5.4) and (2.11) lead to

$$G_1^{(1)}(b, \zeta) = i\sigma^{-1}(i\kappa^2 - \zeta^2)^{1/2}[k^2(1 + C_1) - \zeta^2]^{-1}. \tag{5.6}$$

This may be factored by inspection to give

$$L_+^{(1)}(\zeta) = (i/\sigma)^{1/2}(e^{i\pi/4}\kappa + \zeta)^{1/2}[k(1 + C_1)^{1/2} + \zeta]^{-1} \tag{5.7a}$$

and

$$L_-^{(1)}(\zeta) = (i/\sigma)^{-1/2}(e^{i\pi/4}\kappa - \zeta)^{-1/2}[k(1 + C_1)^{1/2} - \zeta]. \tag{5.7b}$$

In particular, it follows from (3.9) and (3.8) respectively that

$$C^{(1)} = -\kappa^{1/2}\sigma^{-1/2}e^{-i\pi/8}[1 - C_1/4] \tag{5.8}$$

and

$$\mathcal{E}_z^{(1)}(b, \zeta) = -e^{i\pi/8}\kappa^{1/2}\sigma^{-1}[1 - C_1/4][e^{i\pi/4}\kappa - \zeta]^{1/2}[k(1 + C_1)^{1/2} - \zeta]^{-1}. \tag{5.9}$$

Note that $\mathcal{E}_z$ is proportional to σ^{-1} as expected.

The advantage of this approximation is that the factorization can be carried out explicitly. However, the exact Green's function $G_0(b, \zeta)$ as given by (2.3) has branch points at $\zeta = \pm k$, while the function $G_0^{(1)}(b, \zeta)$ as given by (5.4) has no branch points there. Since it is precisely these branch points that determine the behavior of the field vectors for large z, this first approximation is not good enough. Since, as $\xi \to 0$, the series expansion of the Hankel function yields that $| H_0^{(1)}(\zeta c)/H_0^{(1)\prime}(\xi c) | = O(\xi \ln \xi)$, a better approximation of Q_2 in (2.5) is

$$Q_2^{(2)} = i(1 + P), \tag{5.10}$$

where

$$P = 4H_0^{(1)}(\xi c)e^{2i\eta(c-b)}[\eta\xi^{-1}(k/\kappa)^2 H_0^{(1)\prime}(\xi c) - H_0^{(1)}(\xi c)]. \tag{5.11}$$

The term P is important only when ξ is small. Therefore it is sufficient to use

$$L_+^{(2)}(k) = L_+^{(1)}(k), \quad \text{or} \quad C^{(2)} = C^{(1)}. \tag{5.12}$$

Let

$$M(\zeta) = P[1 + C_1(k/\xi)^2]^{-1}, \tag{5.13}$$

then (5.10) leads to

$$G_1^{(2)}(b, \zeta) = G_1^{(1)}(b, \zeta)[1 + M(\zeta)]^{-1}. \tag{5.14}$$

Let the quantity in the bracket of (5.14) be factored in the form

$$[1 + M(\zeta)]^{-1} = [1 + N_+(\zeta)]/[1 + N_-(\zeta)], \tag{5.15}$$

where

$$\begin{aligned} N_\pm(\zeta) &= -1 + \exp\left\{-(2\pi i)^{-1}\int_{-\infty \mp i\epsilon/2}^{\infty \mp i\epsilon/2} dt(t - \zeta)^{-1}\ln[1 + M(t)]\right\} \\ &= -(2\pi i)^{-1}\int_{-\infty \mp i\epsilon/2}^{\infty \mp i\epsilon/2} dt(t - \zeta)^{-1}M(t). \end{aligned} \tag{5.16}$$

It follows from (5.15) that

$$L_{-}^{(2)}(\zeta) = L_{-}^{(1)}(\zeta)[1 + N_{-}(\zeta)]. \tag{5.17}$$

In the next few sections, the electromagnetic field at various points of the space will be studied. For simplicity all superscripts (1) and (2) will be omitted and only leading terms retained since, except at great distances from the junction at $z = 0$, the first approximation and the second approximation differ negligibly.

6. Approximate field inside the coaxial line. In this section, the behavior of the field for $a \leq r \leq b$ is considered. For $a \leq r \leq b$, let

$$S_1(r, \zeta) = \frac{H_0^{(1)}(\xi a)H_0^{(2)}(\xi r) - H_0^{(2)}(\xi a)H_0^{(1)}(\xi r)}{H_0^{(1)}(\xi a)H_0^{(2)}(\xi b) - H_0^{(2)}(\xi a)H_0^{(1)}(\xi b)}, \tag{6.1}$$

so that

$$\mathcal{E}_z(r, \zeta) = \mathcal{E}_z(b, \zeta)S_1(r, \zeta). \tag{6.2}$$

Note that as $\xi \to 0$,

$$S_1(r, \zeta) \sim \ln (r/a)/\ln (b/a). \tag{6.3}$$

Equation (6.3) implies that, when $z \gg b$, the transverse distribution of E_z is essentially $\ln(r/a)$.

The region $a \leq r \leq b$ may be divided into various subregions I $-$ V as follows

Region I: $-z \gg b$;

Region III: $z \gg b$ but $kz \mid C_1 \mid \ll 1$, and

Region V: $k \mid C_1 \mid z \gg 1$.

Region II is the region between regions III and I; and

Region IV is the region between regions III and V.

The region II is the junction region and will not be studied further.

Let region I be considered first. In this region, E_z is obviously very small. Thus, it is necessary to consider $H_\theta(r, z)$. It follows from (6.2) that the Fourier transform of $H_\theta(r, z)$ is

$$\mathcal{H}_\theta(r, \zeta) = i\omega\epsilon\xi^{-2}\mathcal{E}_z(b, \zeta)(\partial/\partial r)S_1(r, \zeta). \tag{6.4}$$

If Res denotes "residue of", then in this region (3.1) holds with

$$\Gamma = -[\operatorname*{Res}_{\zeta=-k} \xi^{-2}\mathcal{E}_z(b, \zeta)]/[\operatorname*{Res}_{\zeta=k} \xi^{-2}\mathcal{E}_z(b, \zeta)] = C_1/4. \tag{6.5}$$

Thus

$$H_\theta(r, z) = \frac{r}{b}\,[e^{ikz} + \tfrac{1}{4}C_1e^{-ikz}]. \tag{6.6}$$

For region III, it follows from (5.9), (6.2) and (6.3) that

$$E_z(r, z) = e^{-i\pi/4}\kappa\sigma^{-1}\,\frac{\ln (r/a)}{\ln (b/a)}\,e^{ikz}\exp\left[e^{i3\pi/4}\,\frac{kz}{\kappa b\ln (b/a)}\right], \tag{6.7}$$

and

$$H_\theta(r, z) = \frac{b}{r}\,e^{ikz}\exp\left[e^{i3\pi/4}\,\frac{kz}{\kappa b\ln (b/a)}\right]. \tag{6.8}$$

Therefore, the conventional theory of waveguides with a slightly lossy wall is valid provided that $kz \mid C_1 \mid \ll 1$ where C_1 is defined in (5.5).

For the region V, the second approximation has to be used. The substitution of (5.17) into (3.8) gives

$$F^{(2)}(\zeta) = F^{(1)}(\zeta)[1 - N_-(\zeta)]. \tag{6.9}$$

When z is very large, only a small region in the neighborhood of $\zeta = k$ can contribute to the inverse Fourier transform. Therefore, it is only necessary to consider the second term $-F^{(1)}(\zeta)N_-(\zeta)$, since $F^{(1)}(\zeta)$ is analytic at $\zeta = k$. With this notion, $N_-(\zeta)$ may be found explicitly near $\zeta = k$ as follows:

$$N_-(\zeta) = M(\zeta), \tag{6.10}$$

since $N_+(\zeta)$ is analytic at $\zeta = k$. On the other hand, it follows from (5.13) that, near $\zeta = k$,

$$M(\zeta) = \xi^4[k^2(1 + C_1) - \zeta^2)]^{-1}4\kappa ck^{-2}e^{-i\pi/4} \ln (\xi c) \exp [2i\eta_0(c - b)], \tag{6.11}$$

where $\eta_0 = \kappa \exp (i\pi/4)$. Consequently, for this region

$$\begin{aligned} F^{(2)}(\zeta) &= -F^{(1)}(\zeta)M(\zeta) \\ &= 2\kappa^2k^{-3}\sigma^{-1}[k(1 + C_1/2) - \zeta]^{-2} \xi^4c \ln (\xi c) \exp [2i\eta_0(c - b)]. \end{aligned} \tag{6.12}$$

The substitution of (6.3) and (6.12) into (6.2) now yields

$$E_z(r, z) = \frac{c}{\pi} \kappa^2k^{-3}\sigma^{-1} \exp [2i\eta_0(c - b)] \frac{\ln (r/a)}{\ln (b/a)} F, \tag{6.13}$$

where

$$F = \int_{-\infty}^{\infty} d\zeta e^{i\zeta z}[k(1 + C_1/2) - \zeta]^{-2}(k^2 - \zeta^2)^2 \ln [(k^2 - \zeta^2)^{1/2}c]. \tag{6.14}$$

The integral in (6.14) may be evaluated by closing the contour of integration in the upper half plane. Thus

$$\begin{aligned} F = -4\pi iC_1^2k^3[1 - 2\pi i + 2 \ln (k^2C_1c^2)] - 2\pi zC_1^2k^4[\ln (k^2C_1c^2) - i\pi] \\ + 4\pi ik^2e^{ikz} \int_0^{\infty} d\zeta e^{i\zeta z}\left[\frac{kC_1}{2} - \zeta\right]^{-2}\zeta^2. \end{aligned} \tag{6.15}$$

The first two terms on the right hand of (6.15) are never large enough to make a significant contribution so that they may be omitted. Thus

$$E_z(r, z) = 4ic\kappa^2k^{-1}\sigma^{-1}e^{ikz} \{\exp [2i\eta_0(c - b)]\} \frac{\ln (r/a)}{\ln (b/a)} \int_0^{\infty} d\zeta e^{i\zeta z}\left[\frac{kC_1}{2} - \zeta\right]^{-2}\zeta^2. \tag{6.16}$$

As $z \to \infty$ a stationary phase integration yields

$$E_z(r, z) \sim -32i\kappa^4cb^2k^{-3}\sigma^{-1}e^{ikz}\{\exp [2i\eta_0(c - b)]\} \ln (b/a) \ln (r/a)z^{-3} \tag{6.17}$$

and

$$H_\theta(r, z) = 8i\kappa^{-2}cb^2k^{-2}e^{ikz}\{\exp [2i\eta_0(c - b)]\} \ln (b/a) \frac{1}{r} z^{-2}. \tag{6.18}$$

Note that E_z behaves like z^{-3}, and hence H_θ and E_r behave like z^{-2}. It is interesting to note that the asymptotic phase velocity is that of vacuum.

Finally, there is still region IV to be studied. Because of the form of Eq. (6.9), it is only necessary to add (6.7) and (6.16). Thus

$$E_z(r, z) = \left\{e^{-i\pi/4} \exp\left(\frac{e^{i3\pi/4}kz}{\kappa b \ln(b/a)}\right) + 4ic\kappa k^{-1}\{\exp[2i\eta_0(c - b)]\} \int_0^\infty d\zeta e^{i\zeta z}\left[\frac{kC_1}{2} - \zeta\right]^{-2}\zeta^2\right\}\kappa\sigma^{-1}e^{ikz}\, \frac{\ln(r/a)}{\ln(b/a)}. \tag{6.19}$$

This is an approximate formula valid for all z so that $z \gg b$. It gives explicitly the correction to the simple theory (given by the first term) for waveguides with leaky walls.

These explicit results varify in detail the discussion at the beginning of Sec. 4.

7. Approximate field in the imperfect conductor. The region occupied by the imperfect conductor may also be divided into five subregions, I′ − V′, as given in the last section except that here $b \leq r \leq c$. In region III′, the conventional waveguide theory with exponential dependence of z must hold again. Thus attention is concentrated on the region V′.

The function that corresponds to the S_1 of (6.1) is now

$$S_2(r, \zeta) = D(r)/D(b), \tag{7.1}$$

where

$$D(r) = \begin{vmatrix} H_0^{(1)}(\xi c) & H_0^{(1)}(\eta r)H_0^{(2)}(\eta c) - H_0^{(2)}(\eta r)H_0^{(1)}(\eta c) \\ \eta\xi^{-1}k^2(k^2 + i\kappa^2)^{-1}H_0^{(1)\prime}(\xi c) & H_0^{(1)}(\eta r)H_0^{(2)\prime}(\eta c) - H_0^{(2)}(\eta r)H_0^{(1)\prime}(\eta c) \end{vmatrix}. \tag{7.2}$$

Analogous to (6.2), it follows that, for $b \leq r \leq c$,

$$\mathcal{E}_z(r, \zeta) = \mathcal{E}_z(b, \zeta)S_2(r, \zeta). \tag{7.3}$$

For the present purpose, the appropriate approximate formula for S_2 is

$$S_2(r, \zeta) = (b/r)^{1/2}\{\exp[i\eta_0(r - b)]\}\big(1 - \exp[2i\eta_0(c - r)] - 2e^{-i\pi/4}\{\exp[2i\eta_0(c - r)] - \exp[2i\eta_0(c - b)]\}\kappa c(\xi/k)^2 \ln(\xi c)\big) \tag{7.4}$$

when ξ/k is small. With (7.3) written in the form

$$\mathcal{E}_z(r, \zeta) = \mathcal{E}_z^{(1)}(b, \zeta)[1 - M(\zeta)]S_2(r, \zeta), \tag{7.5}$$

the electromagnetic field in region V′ is found to be

$$E_z(r, z) = 4\kappa^{-3}cb\sigma^{-1}k^{-3}(b/r)^{1/2} \ln(b/a)e^{ikz} \exp[i\eta_0(r - b)]\big(e^{i\pi/4}k\{\exp[2i\eta_0(c - r)] - \exp[2i\eta_0(c - b)]\}z^{-2} - 8i\kappa b \ln(b/a) \exp[2i\eta_0(c - b)]z^{-3}\big), \tag{7.6}$$

and, with the second term in the parentheses neglected

$$H_z(r, z) = 4i\kappa^2cbk^{-2}(b/r)^{1/2} \ln(b/a)e^{ikz}\{\exp[i\eta_0(r - b)]\}z^{-2} \cdot\{\exp[i\eta_0(c - r)] + \exp[i\eta_0(c + r - 2b)]\}. \tag{7.7}$$

Here the second term in the brackets evidently represents a reflection at $r = b$. It is noted that (7.6) and (6.17) give the same value for E_z at $r = b$, while (7.7) and (6.18)

give the same value for H_θ at $r = b$. This should be the case for a self-consistent scheme of approximation.

A quanity of particular interest is the transverse distribution of the magnetic field in the outer conductor. For region III′, it is

$$| H_\theta(r, z) | \sim r^{-1/2}\left[\exp\frac{-(r-b)}{\delta} + \exp\frac{-(2c-b-r)}{\delta}\right], \tag{7.8}$$

while for region V′, it is

$$| H_\theta(r, z) | \sim r^{-1/2}\left[\exp\frac{-(c-r)}{\delta} + \exp\frac{-(c+r-2b)}{\delta}\right]. \tag{7.9}$$

These distributions indicate clearly that, for regions III and III′, the electromagnetic energy effectively leaks from the coaxial line to the radiation field, while for regions V and V′, the electromagnetic energy effectively leaks from the radiation field into the coaxial line. This lends further support to the qualitative discussion of Sec. 4. For region IV′, the distribution of the magnetic field must be intermediate between that given by (7.8) and that of (7.9). These results are schematically sketched in Fig. 2.

8. Approximate radiation field for large z. It remains to consider the situation outside of the coaxial line where the structure of the field is enormously complicated. In this section, attention is restricted to the extension of regions V and V′, namely the region V″: $kz \mid C_1 \mid \gg 1$ and $r > c$. This part of the calculation is therefore a continuation of that of the last section. In the next section the radiation pattern is to be found approximately, as a continuation of Sec. 4. In Sec. 10, attention is turned to the extension III″ of regions III and III′. There the relation with the lowest "mode" of the coaxial line is evident, and the present theory puts a limit on the range of validity of this "mode" solution.

For $r > c$ and according to (2.3), the function that corresponds to S_1 and S_2 is given by

$$S_3(r, \zeta) = \frac{4H_0^{(1)}(\xi r)}{\pi i \eta c\, D(b)}, \tag{8.1}$$

and E_z is determined by

$$\mathcal{E}_z(r, \zeta) = \mathcal{E}_z(b, \zeta) S_3(r, \zeta). \tag{8.2}$$

The situation here differs from the previous ones in that $H_0^{(1)}(\xi r)$, which is a factor in

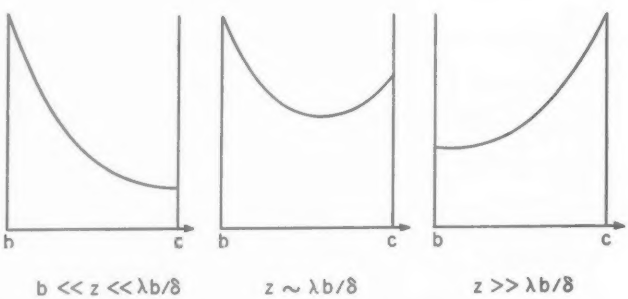

FIG. 2. Sketches of magnetic field distribution in the outer conductor.

(8.1), has branch points at $\zeta = \pm k$. Furthermore, other contributions with branch points, from the determinant and from $F(\zeta)$, have at least a factor ξ^2. For region V″, these other contributions may be neglected. Therefore, (5.9) is adequate for use in (8.2) with the result

$$\mathcal{E}_z(r, \zeta) = -2\pi e^{i\pi/4}\sigma^{-1}\kappa^3 k^{-3} b^{3/2} c^{1/2} \xi^2 \ln (b/a)\{\exp [i\eta_0(c - b)]\} H_0^{(1)}(\xi r). \tag{8.3}$$

From the discontinuity of the Hankel function across the branch cut, the inverse Fourier transform gives

$$E_z(r, z) = -2e^{i\pi/4}\sigma^{-1}\kappa^3 k^{-3} b^{3/2} c^{1/2} \ln (b/a)\{\exp [i\eta_0(c - b)]\} e^{ikz} r^{-4} B\left(\frac{z}{2kr^2}\right), \tag{8.4}$$

where

$$B(x) = \int_0^\infty s^3 \, ds J_0(s) \exp (-is^2 x). \tag{8.5}$$

It does not seem possible to evaluate $B(x)$ explicitly in terms of known functions. However, it is easy to find that, as $x \to \infty$

$$B(x) \sim -\tfrac{1}{2} x^{-2}. \tag{8.6}$$

The substitution of (8.6) into (8.4) yields

$$E_z(r, z) = 4e^{i\pi/4}\sigma^{-1}\kappa^3 k^{-2} b^{3/2} c^{1/2} \ln (b/a)\{\exp [i\eta_0(c - b)]\} e^{ikz} z^{-2}. \tag{8.7}$$

This result is valid for $z \gg b/k\delta$ and $c \le r \ll (z/k)^{1/2}$. Note that this field is independent of r, as must be the case for a far-sone radiation field. Also note that (8.7) is consistent with (7.6).

It is not possible to find $H_\theta(r, z)$ directly from (8.7). To find H_θ, it is necessary to go back to (8.1), the reason being that $\partial/\partial r$ nullifies the leading term in the present calculation. The determination of the magnetic field is not given here since it does not lead to anything interesting.

9. The approximate radiation pattern. In this section, (4.3) is to be simplified under the assumptions (5.1) and (5.2). First, the functions $L_+(k)$ and $L_-(k \cos \phi)$ may be eliminated by (3.8), (3.9) and (5.9). The rest of the calculation is straightforward with the result

$$\begin{aligned} A(\phi) = {} & k^2\kappa^{-2} b^{1/2} c^{1/2}\{\exp [i\eta_0(c - b)]\}[(1 + C_1)^{1/2} - \cos \phi]^{-1} \sin \phi \\ & \cdot \left[1 - e^{-i\pi/4}\kappa c \sin^2 \phi \ln \frac{\gamma k c \sin \phi}{2} + \frac{\pi}{2} e^{i\pi/4}\kappa c \sin^2 \phi\right]^{-1}, \end{aligned} \tag{9.1}$$

where $\gamma = 1.78107$. Note that $A(0) = 0$, as it should.

It is interesting to know the direction ϕ_0 of the major lobe for this radiation pattern. This may be found as follows. From the last factor in (9.1), it is seen that

$$\kappa c \sin^2 \phi_0 \ln \frac{\gamma k c \sin \phi_0}{2} = O(1). \tag{9.2}$$

Therefore, in the vicinity of ϕ_0, $A(\phi)$ is approximately proportional to

$$A(\phi) \sim \text{const} \sin \phi \left[1 - e^{-i\pi/4}\kappa c \sin^2 \phi \ln \frac{\gamma k c \sin \phi}{2}\right]^{-1}. \tag{9.3}$$

The absolute value of this is

$$| A(\phi) |^2 \sim \text{const} \sin^2 \phi \left[1 - \sqrt{2}\, \kappa c \sin^2 \phi \ln \frac{\gamma k c \sin \phi}{2} + \kappa^2 c^2 \sin^4 \phi \left(\ln \frac{\gamma k c \sin \phi}{2} \right)^2 \right]^{-1}. \tag{9.4}$$

This quantity on the right has a maximum at approximately

$$\phi_0 = \left(\frac{1}{2} \kappa c \ln \frac{\kappa}{k^2 c} \right)^{-1/2}, \tag{9.5}$$

whence

$$| A(\phi_0) | = \kappa^{-1} k^2 b^{3/2} c^{1/2} \left[\exp \frac{-(c-b)}{\delta} \right] \ln (b/a)[1 + 2^{-1/2}]^{1/2}. \tag{9.6}$$

10. The "mode." Finally, consider the region III″ defined by $z \gg b$, $kz \mid C_1 \mid \ll 1$ and $r > c$. For this region (5.9) may be used for $F(\zeta)$. Therefore, from (8.1) and (8.2) it follows that

$$\mathcal{E}_z(r, \zeta) = -e^{i\pi/4} \kappa \sigma^{-1} [k(1 + C_1)^{1/2} - \zeta]^{-1} \frac{4 H_0^{(1)}(\xi r)}{\pi i \eta c} \left[H_0^{(1)}(\eta b) \begin{vmatrix} H_0^{(1)}(\xi c) & H_0^{(2)}(\eta c) \\ \dfrac{\eta}{\xi} \dfrac{k^2}{k^2 + i\kappa^2} H_0^{(1)\prime}(\xi c) & H_0^{(2)\prime}(\eta c) \end{vmatrix} \right]^{-1}, \tag{10.1}$$

and hence

$$\mathcal{H}_z(r, \zeta) = 2 e^{i\pi/4} k^2 \kappa^{-1} \xi^{-1} [k(1 + C_1)^{1/2} - \zeta]^{-1} (b/c)^{1/2} H_1^{(1)}(\xi r) \{ \exp [i \eta_0 (c - b)] \} \cdot \left\{ -i + \frac{2}{\pi} \ln \frac{\gamma \xi c}{2} - \frac{2}{\pi} \frac{k^2 \eta_0}{\kappa^2 \xi^2 c} \right\}^{-1}. \tag{10.2}$$

In particular, when $kr \ll 1$, $\mathcal{H}_z(r, \zeta)$ and hence $H_z(r, \zeta)$ are proportional to $1/r$. This fact agrees with the result given for the lowest "mode." Secondly, when r increases, the phase of $H_1^{(1)}(\xi r)$ cannot decrease near $\zeta = k$, and hence the phase of H_z increases. This again agrees with the result for the lowest "mode" [2]. However, beyond that, the result for the lowest "mode" does not agree quantitatively with (10.2). In other words, for the present method of driving, the lowest "mode" does not dominate anywhere outside of the coaxial line. The "mode" picture therefore has very limited scope of application indeed.

11. Summary and conclusions. The problem of the imperfectly conducting coaxial line driven as shown in Fig. 1 has been studied in some detail. The results for small skin depth may be summarized qualitatively as follows. The place where more detail may be found is written in square brackets.

A. The reflection coefficient in the line at $z = 0$ is of the order $\delta / b \ln(b/a)$ [(6.5)].

B. For $r < c$ and $b \ll z \ll \lambda b/\delta$ the electromagnetic field decays exponentially [(6.7) and (6.8)].

C. In the dielectric inside the coaxial line, i.e., $a \leq r \leq b$, and for $z \gg \lambda b/\delta$, the longitudinal field decays as z^{-3} but the transverse fields decay as z^{-2} [(6.17) and (6.18)].

D. In the outer conductor of the coaxial line and for $z \gg \lambda b/\delta$, all field components decay as z^{-2} [(7.6) and (7.7)].

E. In the dielectric inside the coaxial line, the transverse distribution of each field component is essentially independent of z for all $z \gg b$[(6.19)].

F. In the outer conductor the transverse distributions are not even approximately independent of z [(7.8), (7.9) and Fig. 2].

G. Outside the coaxial line for $z \gg \lambda b/\delta$ and $c \leq r \ll (z/k)^{1/2}$, the field component E_z is approximately independent of r[(8.7)].

H. The radiation pattern has a maximum in a direction slightly inclined from the direction of the coaxial line [(9.5) and (9.6)].

I. The "mode" picture has only qualitative meaning outside of the coaxial line [Sec. 10].

It is to be expected that many of these statements are true for any waveguide with a leaky wall and driven in a similar fashion. Let D_1 be the region enclosed by the waveguide wall, D_2 be the region occupied by the waveguide wall, and $D_3 = D_1 + D_2$. Also let L be a typical transverse dimension of the waveguide, then the following generalizations may be conjectured.

B'. The electromagnetic field decays exponentially for $L \ll z \ll \lambda L/\delta$ in the region D_3.

C'. Assume $z \gg \lambda L/\delta$. Then for a *TEM* incident wave, the longitudinal field decays as z^{-3} but the transverse fields decay as z^{-2} in the region D_1. For a non-*TEM* incident wave, all components decay as z^{-2}.

D'. For $z \gg \lambda L/\delta$ and in the region D_2, all field components decay as z^{-2}.

E' In the region D_1, the transverse distribution of the field components are essentially the same for all $z \gg L$.

F'. The statement in *E'* is not true for D_2.

H'. The radiation pattern has a maximum in a direction slightly inclined from the direction of the waveguide. (This seems to be true for both fast and slow waves.)

I'. The "mode" picture has very limited validity in general.

The validity of these conjectures remains to be investigated. In particular, their correctness for a non-*TEM* incident wave should not be taken for granted.

Acknowledgment. The author is indebted to Professors Ronald W. P. King and George F. Carrier for many discussions of this problem and their correction of the manuscript.

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BOOK REVIEWS

The theory of storage. By P. A. P. Moran. Methuen Co., London, and John Wiley & Sons, Inc., New York, 1959. 111 pp. $2.50.

This little book, the first in a new series of "Methuen Monographs on Applied Probability and Statistics" presents on 105 pages a remarkably compact discussion of what is usually regarded as a rather rambling field: the mathematical theory of inventory control and water storage. In this country most attention has been focused on the first variant of this problem, the management of stocks of repair parts or other items with probabilistic demand so as to maximize some economic criterion of efficiency. This book offers a useful complement in that the dam interpretation of the problem, and those aspects of inventory control that bear most directly on water storage—on which the author is a leading authority—receive the most detailed treatment.

An introductory chapter presents some probability concepts; but the reader should have some knowledge of the probability calculus beforehand. The chapter on the inventory problem emphasizes the calculation of probabilities of states, such as shortages, but pays little regard to the discovery and calculation of optimal policies. A useful and interesting feature is the systematic discussion of various inventory problems that do not involve stochastic processes but can be disposed of as one-period problems. Segerdahl's theory of insurance risk is also presented in outline.

Throughout, the discussion is in terms of specific problems, and solutions are obtained for particular distributions, typically those of the Gamma family. Continuous time problems are analyzed by going to the limit with discrete approximations. The Monte Carlo or simulation method is used to deal with the complicated problems that arise in the study of sequences of dams.

In spite of its great variety of topics the book cannot achieve an adequate survey in its limited space. However, it is an excellent introduction on an elementary level to a field of operations research where much work is going on at the present.

MARTIN BECKMANN

Handbook of supersonic aerodynamics—mechanics of rarefied gases. By Samuel A. Schaaf and Lawrence Talbot. Johns Hopkins University Applied Physics Laboratory, Silver Spring, Maryland, 1959. ii + 85 pp. $1.25.

The present section of the Handbook of Supersonic Aerodynamics is part of a series, the first one of which was issued in 1950. This section deals, as the title indicates, with rarefied gas flows. It opens with a general survey of rarefied flow regimes. Unfortunately, the original description of rarefied flow regimes as put forward by Tsien is still propagated. Following this is a useful chapter on free molecule flow calculations, in which the aerodynamic and heat transfer coefficients for some bodies of simple geometry are presented. The following chapter presents a discussion of slip flow which this reviewer considers to be somewhat dated in the light of more recent advances. This little compendium does have a considerable amount of information, much of it in graphical form, that may be useful to those interested in a cursory view of the field.

RONALD F. PROBSTEIN

German-English mathematics dictionary. Compiled and edited by Charles Hyman. Interlanguage Dictionaries Publishing Co., New York, 1960. 131 pp. $8.00.

This useful dictionary contains more than 8500 entries from applied as well as pure mathematics. Extensive sampling did not reveal any incorrect entries. On the other hand, the temptation to increase the number of entries in a trivial manner has not always been resisted. For example, all of the entries "Axiom, Axiomatik, axiomatisch, axiomatisieren, Axiomatisierung, Axiomensystem" could well have been omitted, and the space gained could have been used to list less trivial correspondences, for instance "Drall—moment of momentum" or "Wirbelmoment—vortex strength."

W. PRAGER

(*Continued on p. 38*)

ADDITION THEOREMS FOR SPHERICAL WAVE FUNCTIONS*

BY

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Abstract. Addition theorems are described for spherical vector wave functions, under both rotations and translations of the coordinate system. These functions are the characteristic solutions in spherical coordinates of the vector wave equation, such as occurs in electromagnetic problems. The vector wave function addition theorems are based on corresponding theorems for the spherical scalar wave functions. The latter are reviewed and discussed.

1. Introduction. Boundary-value problems often involve several bodies or more than one important reference point. It can then be very convenient to be able to expand the field solutions in alternative forms, each form referring to a different specific coordinate set describing the same space. The connections between the alternate representations are provided by "addition theorems", i.e., formulas for the expansion of the basis set of one representation in terms of the basis set of another. In general, the completeness of such basis sets indicates immediately the existence of these expansions, although their specification may not be obvious.

The results discussed here were motivated by a problem in electromagnetic scattering in which exactly such a need arose. The basis sets considered are the so-called spherical vector wave functions, i.e., characteristic *vector* field solutions of the vector wave equation, based on a conventional separation of variables in spherical coordinates (R, θ, ϕ). The set of vector wave functions can be described by using as potentials the set of scalar wave functions (the characteristic solutions to the scalar wave equation). Addition theorems for the latter are however available in the literature. In this paper, these are cited and briefly discussed, and extended to provide the desired general addition theorems for vector wave functions. The only other known calculations of these addition theorems, derived for specific low orders of vector wave functions by direct but laborious procedures, and applied to some particular problems, appears in Ament [1].

2. Characterization of the vector wave functions. The electric and magnetic fields in a source-free homogeneous medium are divergenceless, and each satisfies a vector wave equation of form

$$\nabla \times \nabla \times \mathbf{A} - k^2\mathbf{A} = 0, \tag{1}$$

k being a constant for the medium. It is well-known (Stratton, [2]) that independent solutions of this equation can be constructed as

$$\mathbf{M}_{mn} = \nabla u_{mn} \times \mathbf{R},$$
$$\mathbf{N}_{mn} = \frac{1}{k} \nabla \times \mathbf{M}_{mn}\ , \tag{2}$$

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with the added interrelation $\mathbf{M}_{mn} = 1/k \nabla \times \mathbf{N}_{mn}$, where $\mathbf{R}$ is a position vector from the origin 0, and the potentials u_{mn} are a corresponding complete set of solutions of the *scalar* wave equations

$$\nabla^2 u + k^2 u = 0. \tag{3}$$

In spherical coordinates, such a set of characteristic solutions are given by

$$u_{mn} = z_n(kR)P_n^m(\cos\theta)\exp(im\phi), \quad -n \leq m \leq n, \qquad 0 \leq n < \infty. \tag{4}$$

Here $z_n(kR)$ is generically any spherical Bessel function, say as defined by Stratton 3]. The associated Legendre function $P_n^m(\cos\theta)$ will also be taken following Stratton [4], as

$$P_n^m(x) = \frac{(1 - x^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{dx^{n+m}}(x^2 - 1)^n, \qquad |m| \leq n. \tag{5}$$

The latter differs by a $(-)^m$ factor from one common alternative definition.

The specific forms for the vector wave functions are

$$\mathbf{M}_{mn} = z_n(kR)\left[\mathbf{i}_2 \frac{imP_n^m(\cos\theta)}{\sin\theta}\exp(im\phi) - \mathbf{i}_3\exp(im\phi)\frac{d}{d\theta}P_n^m(\cos\theta)\right], \tag{6a}$$

$$\begin{aligned}\mathbf{N}_{mn} = {}& \mathbf{i}_1 \frac{z_n(kR)}{kR} n(n+1)P_n^m(\cos\theta)\exp(im\phi) \\ & + \frac{1}{kR}\frac{d}{dR}[Rz_n(kR)]\left[\mathbf{i}_2\exp(im\phi)\frac{d}{d\theta}P_n^m(\cos\theta) + \mathbf{i}_3\frac{imP_n^m(\cos\theta)}{\sin\theta}\exp(im\phi)\right],\end{aligned} \tag{6b}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are unit vectors in the directions of increasing R, θ, ϕ respectively. It is useful to note that for $n = m = 0$, we have $\mathbf{M}_{00} = \mathbf{N}_{00} = 0$.

We note that Maxwell's equations for harmonic time-dependence $\exp(-i\omega t)$, and for sourceless regions are

$$\begin{aligned}\nabla \times \mathbf{E} &= i\omega\mu\mathbf{H}, \\ \nabla \times \mathbf{H} &= -i\omega\epsilon\mathbf{E},\end{aligned} \tag{7}$$

where $\omega^2\mu\epsilon \equiv k^2$. It follows from Eq. (2) that to an E-field contribution of the form $\mathbf{M}_{mn}$ there is associated exactly an H-field term, $(k/i\omega\mu)\mathbf{N}_{mn}$; similarly, to an E-field contribution $\mathbf{N}_{mn}$, there is associated an H-field term $(k/i\omega\mu)M_{mn}$. Furthermore, the detailed forms in Eqs. (6) show that $\mathbf{M}_{mn}$ has no radial component, and hence all radial field components must be represented by the $\mathbf{N}_{mn}$ solely. Thus, in fact, it is common to distinguish in electromagnetic theory two types of modes: the H-type in which only the magnetic field has a radial component, and the E-type in which only the electric field has a radial component. It is clear that H-type modes have $\mathbf{E}$ represented by the $\mathbf{M}_{mn}$ and $\mathbf{H}$ by the $\mathbf{N}_{mn}$, and the E-type modes vice-versa. For any general excitation, both types of modes may be present, and so the general field summations would be of the form

$$\mathbf{E} = \sum_{n=1}^{\infty}\sum_{m=-n}^{\infty}[A_{mn}\mathbf{M}_{mn} + B_{mn}\mathbf{N}_{mn}], \tag{8a}$$

$$\mathbf{H} = (k/i\omega\mu)\sum_{n=1}^{\infty}\sum_{m=-n}^{\infty}[A_{mn}\mathbf{N}_{mn} + B_{mn}\mathbf{M}_{mn}]. \tag{8b}$$

Finally we note that if we make either a translation or rotation, or both, of the coordinates, a new set of scalar and vector wave functions can be analogously defined

with respect to the new coordinates. Since any one of the previous vector wave functions defines a perfectly valid vector field, it must be expandable over the new set. That is, under coordinates rotations and/or translations, there *must* exist addition theorems of the form

$$\mathbf{M}_{\mu\nu}(R, \theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [C(\mu, \nu \mid m, n)\mathbf{M}_{mn}(R', \theta', \phi') + D(\mu, \nu \mid m, n)\mathbf{N}_{mn}(R', \theta', \phi')], \tag{9a}$$

$$\mathbf{N}_{\mu\nu}(R, \theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [C(\mu, \nu \mid m, n)\mathbf{N}_{mn}(R', \theta', \phi') + D(\mu, \nu \mid m, n)\mathbf{M}_{mn}(R', \theta', \phi')], \tag{9b}$$

where (R', θ', ϕ') is the new set of spherical coordinates. In the above, we may note that the second equation follows directly from the first by utilizing the basic definitions in Eqs. (2), and noting additionally that the curl operator is invariant of the choice of coordinate system. We note also that the coefficients will of course contain implicitly the geometric parameters which relate the two coordinate systems.

We will describe the representations for rotations and translations separately. A coordinate change involving both types will be a "multiplication" of the two operations.

3. Rotation of coordinates. It is extremely simple to describe the addition theorems for $\mathbf{M}_{mn}$ and $\mathbf{N}_{mn}$ under coordinate rotations.

We assume a second set of coordinates (R, θ', ϕ') centered at 0, but with reference axes rotated with respect to the original (R, θ, ϕ) system. Note that the radial coordinate is invariant under rotation. In Appendix 1, we characterize such a rotation, and state the known addition theorem for the spherical scalar wave functions, which is in the form*

$$u_{\mu n}(R, \theta, \phi) = \sum_{m=-n}^{n} \beta(\mu, m, n)u'_{mn}(R, \theta', \phi'). \tag{10}$$

But the vector operator ∇ is defined independently of the coordinate system, and due to the common origin, the same $\mathbf{R}$ is the position vector for both. It follows immediately by using Eq. (10) and the basic definitions that the vector wave functions are related by

$$\mathbf{M}_{\mu n}(R, \theta, \phi) = \nabla u_{\mu n}(R, \theta, \phi) \times \mathbf{R} = \sum_{m=-n}^{n} \beta(\mu, m, n)\mathbf{M}'_{mn}(R, \theta', \phi)$$

and this is the complete specification for coordinate rotations, of the general characterization in Eq. (9).

4. Translation of coordinate origin. Suppose a second origin of coordinates is taken at a point 0', whose coordinates are (R_0, θ_0, ϕ_0) with respect to the first. The set of spherical coordinates (R', θ', ϕ') is introduced with respect to 0', such that the polar axis, $\theta' = 0$, and the azimuth axis, $\phi' = 0$, are respectively parallel to the corresponding axes, $\theta = 0$ and $\phi = 0$. This is then a rigid translation of the coordinate system. In Appendix 2, we state the known addition theorem for the spherical scalar wave function under such translation, in the form

*The prime notation on the u'_{mn} is to emphasize the coordinate set with respect to which the potential is defined. We will utilize primes for this purpose throughout; under no circumstances should they be interpreted here as indicating differentiations.

$$u_{\mu\nu}(R, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha(\mu, \nu \mid m, n) u'_{mn}(R', \theta', \phi'). \tag{11}$$

(We note from Appendix 2 that, depending on the value of R relative to R_0, the u'_{mn} may involve a spherical Bessel function $z'_n(kR)$ which is generically not of the same type as the $z_\nu(kR)$ involved in $u_{\mu\nu}$. This distinction will not be important in the derivation below, which involves only the local properties of u'_{mn}, whichever Bessel function type is contained.)

Since the grad operation is invariant of coordinate system, we can immediately write *

$$\begin{aligned} \mathbf{M}_{\mu\nu}(R, \theta, \phi) &= \nabla u_{\mu\nu} \times \mathbf{R} \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha(\mu, \nu \mid m, n)[\nabla u'_{mn} \times \mathbf{R}]. \end{aligned} \tag{12}$$

Our problem therefore is to expand terms of the form $\nabla u'_{mn} \times \mathbf{R}$ in terms of $\mathbf{M}'_{mn}$ and $\mathbf{N}'_{mn}$ vector wave functions.

If $\mathbf{R}_0$ is the position vector of the second origin, 0′, with respect to the first origin, 0, we can further write

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{R}'. \tag{13}$$

But then

$$\nabla u'_{mn} \times \mathbf{R} = (\nabla u'_{mn} \times \mathbf{R}_0) + (\nabla u'_{mn} \times \mathbf{R}') \tag{14}$$

and we immediately recognize the last term as simply

$$\nabla u'_{mn} \times \mathbf{R}' = \mathbf{M}'_{mn}. \tag{15}$$

Hence it remains only to determine an expansion for $\nabla u'_{mn} \times \mathbf{R}_0$.

The simplest approach to the latter has appeared to be to identify first the part which is represented by the $\mathbf{N}'_{mn}$ functions, since this involves matching only the vector components in the $\mathbf{i}'_1$ direction. That is, using Eq. (6b), we seek an expansion of the form

$$\begin{aligned} \mathbf{i}'_1 \cdot \nabla u'_{mn} \times \mathbf{R}_0 &= \sum_{p=1}^{\infty} \sum_{q=-p}^{p} A_{qp} \mathbf{i}'_1 \cdot \mathbf{N}'_{qp}, \\ &= \sum_{p=1}^{\infty} \sum_{q=-p}^{p} A_{qp} \frac{z_p(kR')}{kR'} p(p+1) P_p^q(\cos\theta') \exp(iq\phi'). \end{aligned} \tag{16}$$

It may readily be ascertained that a fixed vector field of magnitude and direction equal to $\mathbf{R}_0$ is represented at the general point (R', θ', ϕ') by components

$$\begin{aligned} \mathbf{R}_0 \cdot \mathbf{i}'_1 &= R_0[\sin\theta_0 \sin\theta' \cos(\phi_0 - \phi') + \cos\theta_0 \cos\theta'], \\ \mathbf{R}_0 \cdot \mathbf{i}'_2 &= R_0[\sin\theta_0 \cos\theta' \cos(\phi_0 - \phi') - \cos\theta_0 \sin\theta'], \\ \mathbf{R}_0 \cdot \mathbf{i}'_3 &= R_0[\sin\theta_0 \sin(\phi_0 - \phi')]. \end{aligned} \tag{17}$$

Most of the derivation then consists in utilizing this result, along with trigonometric identities, and well-known Legendre and Bessel function recurrence relations. The algebraic manipulations are tedious but reproducible, and for the sake of brevity, the algebraic details will be omitted here.

*In writing Eq. (12), we can recall explicitly that we are only interested in the form of expansion for $\nu \geq 1$, $|\mu| \leq \nu$; this is implicit throughout the remainder of the discussion.

For the radial part, comparison of the expansions on the two sides of Eq. (16) yields

$$n = m = 0: \mathbf{i}_1' \cdot \nabla u_{00}' \times \mathbf{R}_0 = 0,$$

$$\begin{aligned} n \geq 1: \mathbf{i}_1' \cdot \nabla u_{mn}' \times \mathbf{R}_0 = \frac{kR_0}{n(n+1)} \mathbf{i}_1' \cdot \Big\{ & im \cos \theta_0 \mathbf{N}_{mn}' \\ & - \frac{i}{2} \sin \theta_0 \exp(-i\phi_0) \mathbf{N}_{m+1,n}' \\ & - \frac{i}{2} \sin \theta_0 \exp(i\phi_0)(n+m)(n-m+1) \mathbf{N}_{m-1,n}' \Big\}. \end{aligned} \tag{18}$$

This completely identifies the radial component, and hence all the possible $\mathbf{N}_{mn}'$ which enter the expansion of $\nabla u_{mn}' \times \mathbf{R}_0$.

If we next consider the $\mathbf{i}_2'$ or $\mathbf{i}_3'$ component of $\nabla u_{mn}' \times \mathbf{R}_0$, subtracting off the corresponding component of the vector whose radial component is indicated on the *RHS* of Eq. (18), we can (after much manipulation) identify the remainder in terms of components of $\mathbf{M}_{mn}'$. The result may be summarized as follows:

$n = m = 0$:

$$\nabla u_{00}' \times \mathbf{R}_0 = kR_0 \cos \theta_0 \mathbf{M}_{01}' + \frac{kR_0}{2} \sin \theta_0 \exp(-i\phi_0) \mathbf{M}_{11}' - kR_0 \sin \theta_0 \exp(i\phi_0) \mathbf{M}_{-1,1}' \tag{19a}$$

and for $n \geq 1$

$$\begin{aligned} \nabla u_{mn}' \times \mathbf{R}_0 - \frac{kR_0}{n(n+1)} \Big\{ & im \cos \theta_0 \mathbf{N}_{mn}' - \frac{i}{2} \sin \theta_0 \exp(-i\phi_0) \mathbf{N}_{m+1,n}' \\ & - \frac{i}{2} \sin \theta_0 \exp(i\phi_0)(n+m)(n-m+1) \mathbf{N}_{m-1,n}' \Big\} \\ = \mathbf{M}_{mn}' + \frac{kR_0}{2n+1} \Big\{ & \cos \theta_0 \Big[\frac{n+m}{n} \mathbf{M}_{m,n-1}' + \frac{n-m+1}{n+1} \mathbf{M}_{m,n+1}' \Big] \\ & + \frac{\sin \theta_0 \exp(-i\phi_0)}{2} \Big[-\frac{1}{n} \mathbf{M}_{m+1,n-1}' + \frac{1}{n+1} \mathbf{M}_{m+1,n+1}' \Big] \\ & + \frac{\sin \theta_0 \exp(i\phi_0)}{2} \Big[\frac{(n+m)(n+m-1)}{n} \mathbf{M}_{m-1,n-1}' \\ & - \frac{(n-m+1)(n-m+2)}{n+1} \mathbf{M}_{m-1,n+1}' \Big] \Big\}. \end{aligned} \tag{19b}$$

We may note that Eq. (19a) is subsumed in Eq. 19b if it is assumed as a definition that $\mathbf{M}_{p,n-1} = 0$ when $n < 0$ and that all terms in $\mathbf{N}_{pn}$ are $\equiv 0$ when $n = 0$; we will use these conventions in the last result below.

We can refer back now to the general form of expansion in Eq. (9), and summarize Eqs. (11, 12, 14, 15 and 19). Further, making some obvious appropriate changes in certain of the summation indices, we find that the addition theorem under rigid coordinate translation has the form

$$\begin{aligned} \mathbf{M}_{\mu\nu}(R, \theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [& C(\mu, \nu \mid m, n) M_{mn}'(R', \theta', \phi') \\ & + D(\mu, \nu \mid m, n) N_{mn}'(R', \theta', \phi')], \end{aligned} \tag{20a}$$

where

$$C(\mu, \nu \mid m, n) = \alpha(\mu, \nu \mid m, n) + \frac{kR_0 \cos \theta_0}{2n+3} \frac{n+m+1}{n+1} \alpha(\mu, \nu \mid m, n+1)$$
$$+ \frac{kR_0 \cos \theta_0}{(2n-1)} \frac{n-m}{n} \alpha(\mu, \nu \mid m, n-1)$$
$$- \frac{kR_0 \sin \theta_0 \exp(-i\phi_0)}{2(2n+3)(n+1)} \alpha(\mu, \nu \mid m-1, n+1)$$
$$+ \frac{kR_0 \sin \theta_0 \exp(-i\phi_0)}{2(2n-1)n} \alpha(\mu, \nu \mid m-1, n-1) \tag{20b}$$
$$+ \frac{kR_0 \sin \theta_0 \exp(i\phi_0)}{2(2n+3)} \frac{(n+m+2)(n+m+1)}{n+1} \alpha(\mu, \nu \mid m+1, n+1)$$
$$- \frac{kR_0 \sin \theta_0 \exp(i\phi_0)}{2(2n-1)} \frac{(n-m-1)(n-m)}{n} \alpha(\mu, \nu \mid m+1, n-1),$$

$$D(\mu, \nu \mid m, n) = \frac{ikR_0 \cos \theta_0}{n(n+1)} m\alpha(\mu, \nu \mid m, n)$$
$$- \frac{ikR_0 \sin \theta_0 \exp(-i\phi_0)}{2n(n+1)} \alpha(\mu, \nu \mid m-1, n) \tag{20c}$$
$$- \frac{ikR_0 \sin \theta_0 \exp(i\phi_0)}{2n(n+1)} (n+m+1)(n-m)\alpha(\mu, \nu \mid m+1, n).$$

In the above $\alpha(\mu, \nu \mid m, n)$ is taken to vanish whenever $\mid m \mid > n$ (see, e.g., Eqs. (A2-4 and A2-2) of Appendix 2).

It would of course be helpful if the expressions for $C(\mu, \nu \mid m, n)$ and $D(\mu, \nu \mid m, n)$ could be reduced, or could be shown to satisfy useful recurrence formulas. One might expect that use of formulas such as indicated in Appendix 2 might lead to such a reduction. However, to date, although some suggestive results have been noted, no clear cut simplifications over the formulas above have been found.

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Appendix I
Addition Theorems for Scalar Spherical Wave Functions Under Coordinate Rotations

These theorems are well known (especially in applications in quantum mechanics) as a special case of the study of the 3-dimensional rotation group. Various forms of the derivations, in largely non-group theoretical terms, appear in [5–9]. Some group theoretical derivations appear in [10], along with general recurrence formulas for the coefficients, and tables covering lower-order cases. We have found it convenient, however, to refer primarily to the forms common in quantum mechanics, such as presented by Edmonds [11]. In terms of the Euler angles of rotation, the addition theorem appears as

$$Y_{jm}(\theta, \phi) = \sum_{m'=-j}^{j} Y_{jm'}(\theta', \phi') \, D^{(j)}_{m'm}(\alpha\beta\gamma), \tag{A1-1}$$

with the relation to our notation [12] of

$$Y_{jm}(\theta, \phi) = (-)^m \left[\left(\frac{2j+1}{4\pi} \right) \frac{(j-m)!}{(j+m)!} \right]^{1/2} P_j^m (\cos \theta) \exp (im\phi). \tag{A1-2}$$

The coordinates θ, ϕ, are the angle coordinates of a point in space defined in the usual manner with respect to one set of cartesian axes, and θ', ϕ' are similarly the coordinates of the same point with respect to a new set of axes, the latter set of axes being obtained by a rigid-body rotation of the first set of axes through the Euler angle α, β, γ. The convention used by Edmonds [13] is that the angles α, β, γ are positive in the sense of rotation of a right-handed screw in the right-handed frame of axes. The coefficients $D^{(j)}_{m'm}$ $(\alpha\beta\gamma)$ are the matrix elements of a unitary transformation, and are given by

$$D^{(j)}_{m'm}(\alpha\beta\gamma) = [\exp im'\alpha]\, d^{(j)}_{m'm}(\beta)[\exp im\gamma], \tag{A1-3}$$

where

$$d^{(j)}_{m'm}(\beta) = \left[\frac{(j+m')!\,(j-m')!}{(j+m)!\,(j-m)!} \right]^{1/2} \sum_\sigma \binom{j+m}{j-m'-\sigma} \binom{j-m}{\sigma} \cdot (-)^{j-m'-\sigma} \left(\cos \frac{\beta}{2} \right)^{2\sigma+m'+m} \left(\sin \frac{\beta}{2} \right)^{2j-2\sigma-m'-m}. \tag{A1-4}$$

In terms of our form (Eq. 10), we note that both sides of Eq. (A1-1) involve surface harmonics of the same order, j, and hence it is a trivial step to introduce additional radial wave function factors $z_j(kR)$ on both sides to obtain a relation on total wave functions. Using our definition of the wave function (Eq. 4) we can rewrite Eq. (A1–1) in the form

$$u_{\mu n}(R, \theta, \phi) = \sum_{m=-n}^{n} \beta(\mu, m, n) u_{mn}(R', \theta', \phi'), \tag{A1-5}$$

where, using also Eq. (A1-2),

$$\beta(\mu, m, n) = (-)^{m+\mu} \left[\frac{(n+\mu)!\,(n-m)!}{(n-\mu)!\,(n+m)!} \right]^{1/2} D^{(n)}_{m\mu}(\alpha\beta\gamma). \tag{A1-6}$$

Several useful sets of recurrence formulas are available for these coefficients, as special cases of general transformation laws [14–17]. but are omitted here, for brevity.

Appendix II
Addition Theorems for the Scalar Spherical Wave Functions Under Coordinate Translations

1. A restatement of the Friedman-Russek result. A derivation of the translational addition theorem for scalar spherical wave functions has been given by Friedman and Russek [18]. However, they use a normalized Legendre function such that P_n^{-m} $(\cos \theta) = P_n^m$ $(\cos \theta)$; and they appear to fail to note that their final statement of the addition theorem depends on the premise that $| m + \mu | = | m | + | \mu |$ where m, μ are azimuthal indices of associated Legendre functions. Since the latter is not correct when m and μ are of opposite sign, we cite here the basically correct form of their result, but give the correct values for the coefficients. In addition Friedman and Russek only identify these coefficients as the values of certain integrals over triple products of Legendre functions, the values of which in turn are found in the literature as rather complicated

sums. Instead, we point out that these coefficients are quite well known in the study of angular momentum in quantum mechanics, being in essence the vector-coupling coefficients (or Wigner $3-j$ symbols) which describe the matrix elements of a unitary transformation between a product-state representation and a compound-state representation in a two-particle system. With this recognition, we are able to draw upon a large body of known information for useful computational forms and recurrence relations. Finally, we will correct below some errors of notation in the original results, as well as present a more useful form for one set of the results.

To begin with, let us recall that our basic definition of the Legendre function (Eq. 5) can be regarded [19] as valid for negative m as well as positive, in the range $-n \leq m \leq n$ (we have in fact so used P_n^m in the text).

With this in mind, the basic result derived by Friedman and Russek in [18] can be stated as follows:

In an (R, θ, ϕ) coordinate system, let the point $0'$ at (R_0, θ_0, ϕ_0) be taken as the origin of a second coordinate system with coordinates (R', θ', ϕ'), and oriented so that a rigid body translation [by the vector $\mathbf{R}_0(R_0, \theta_0, \phi_0)$] takes one system into the other. Then we have the addition theorem,* valid for $\nu \geq 0$, $-\nu \leq \mu \leq \nu$, and where z_ν is any spherical cylinder function,

$$\begin{aligned} z_\nu(kR)P_\nu^\mu(\cos\theta)\exp(i\mu\phi) = i^{-\nu}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{p}\Big\{ & i^{n+p}(2n+1)(-)^m a(\mu, m \mid p, \nu, n) \\ & \cdot z_p(kr_>)P_p^{\mu+m}(\cos\theta_>)\exp[i(\mu+m)\phi_>] \qquad \text{(A2-1)} \\ & \cdot j_n(kr_<)P_n^{-m}(\cos\theta_<)\exp(-im\phi_<)\Big\}, \end{aligned}$$

where

$$\left\{\begin{matrix} r_> = R' & r_< = R_0 \\ \theta_> = \theta' & \theta_< = \theta_0 \\ \phi_> = \phi' & \phi_< = \phi_0 \end{matrix}\right\} \quad \text{when} \quad R' \geq R_0,$$

and

$$\left\{\begin{matrix} r_> = R_0 & r_< = R' \\ \theta_> = \theta_0 & \theta_< = \theta' \\ \phi_> = \phi_0 & \phi_< = \phi' \end{matrix}\right\} \quad \text{when} \quad R' \leq R_0 .$$

The sum over p is over all the values

$$p = n + \nu, \qquad n + \nu - 2, \qquad n + \nu - 4, \cdots$$

(but no lower than $|n - \nu|$), for which the coefficient $a(\mu, m \mid p, \nu, n)$ is non-vanishing. The latter coefficient is defined by the expansion

$$P_n^m(\cos\theta)P_\nu^\mu(\cos\theta) = \sum_p a(\mu\ m, \mid p, \nu, n)P_p^{m+\mu}(\cos\theta) \qquad \text{(A2-2)}$$

*The introduction and use below of the notation $\theta_>$, $\theta_<$, $\phi_>$, $\phi_<$ serves to correct a rather obvious error in the Friedman-Russek paper, in the writing of their Eqs. (19) and (21); the error is obvious from their derivation which involves the parameters only in the quantities $r\cos\gamma$ and $r_0\cos\gamma_0$.

which is known to exist in exactly such a form on the basis of spherical harmonic expansion theorems, with $n + \nu \geq p \geq | n - \nu |$. It is apparent from the orthonormality properties of the p_p^q (cos θ) that $a(\mu, m \mid p, \nu, n)$ can be determined from Eq. (A2-2) in terms of an integral over a special triple product of associated Legendre functions. Such integrals have been investigated in the literature of quantum mechanics, and Friedman and Russek cite one such evaluation, a rather complicated sum involving multitudinous factorials.

Now, it is desirable to rewrite the result, Eq. (A2-1), in a more useful form, namely the form Eq. (11) in which we have used the addition theorem in the text

$$u_{\mu\nu}(R, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha(\mu, \nu \mid m, n) u'_{mn}(R', \theta', \phi'). \tag{A2-3}$$

A comparison with Eq. (A2–1) shows that for $R' \leq R_0$, we have exactly such a form where

$$\alpha(\mu, \nu \mid m, n) = i^{-\nu+n}(2n + 1)(-)^m \sum_p i^p a(\mu, - m \mid p, \nu, n) u_p^{\mu-m}(R_0 , \theta_0 , \phi_0). \tag{A2-4}$$

We note that in these equations for $R' \leq R_0$, the u'_{mn} (R', θ', ϕ') contain explicitly a $j_n(kR')$ dependence, no matter what the form of the spherical cylinder function in $u_{\mu\nu}$ (R, θ, ϕ); and on the other hand, the $u_p^{\mu-m}$ $(R_0 , \theta_0 , \phi_0)$ contains exactly the same type of dependence $z_p(kR_0)$ as is contained in $u_{\mu\nu}$ (R, θ, ϕ) in the form $z_\nu(kR)$.

However, for $R' \geq R_0$, the R' variations in the form of Eq. (A2-1) go into the $u_p^{\mu-m}$, and in order to obtain a form like Eq. (A2-3), it is necessary to interchange the orders of summation. That such an interchanged form must be available is obvious, since the result is then exactly the one expected from the usual technique of expanding an arbitrary function of (R, θ, ϕ) in terms of a new coordinate origin and new coordinate set. We accomplish the interchange by noting that in Eq. (A2-1), p eventually takes on all values between 0 and ∞. Furthermore, the formal extension of the inner sum to $0 < p < \infty$ can be made since the $a(\mu, m \mid p, \nu, n)$ will vanish for all the added terms. We can also extend the sum on m to $(- \infty, \infty)$ since P_n^m vanishes for all the added terms. But now, if we interchange the p- and n-summations, and substitute a new index $t = \mu + m$, we obtain from Eq. (A2-1)

$$\begin{aligned} z_\nu(kR)P_\nu^\mu(\cos \theta) \exp (i\mu\phi) = i^{-\nu} \sum_{p=0}^{\infty} \sum_{t=-\infty}^{\infty} \sum_{n=0}^{\infty} \Big\{ & i^{n+p}(2n + 1)(-)^{t-\mu} a(\mu, t - \mu \mid p, n, \nu) \\ & \cdot z_p(kr_>)P_p^t(\cos \theta_>) \exp (it\phi_>) \\ & \cdot j_n(kr_<)P_n^{\mu-t}(\cos \theta_<) \exp [i(\mu - t)\phi_<]\Big\}. \end{aligned} \tag{A2-5}$$

If we now interchange the letters p and n; write an m for t; and also note that $P_p^t = 0$ whenever $| t | > p$, we can rewrite the above as

$$\begin{aligned} z_\nu(kR)P_\nu^\mu(\cos \theta) \exp (i\mu\phi) = i^{-\nu} \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} \sum_{p=0}^{\infty} \Big\{ & i^{n+p}(2p + 1)(-)^{m-\mu} a(\mu, m - \mu \mid n, p, \nu) \\ & \cdot z_n(kr_>)P_n^m(\cos \theta_>) \exp (im\phi_>) \\ & \cdot j_p(kr_<)P_p^{\mu-m}(\cos \theta_<) \exp [i(\mu - m)\phi_<]\Big\}. \end{aligned} \tag{A2-6}$$

It is readily confirmed that the coefficient in this last equation also vanishes unless, at least, $n + \nu \geq p \geq | n - \nu |$, so that it is even more in a form similar to Eq. (A2-1). By now comparing with Eq. (A2-3), we see that for $R' \geq R_0$, the result corresponding to Eq. (A2-4) is

$$\alpha(\mu, \nu \mid m, n) = i^{-\nu+n}(-)^{m-\mu} \sum_p i^p(2p + 1)a(\mu, m - \mu \mid n, \nu, p)u_p^{\mu-m}(R_0, \theta_0, \phi_0), \quad \text{(A2-7)}$$

where now $u_p^{\mu-m}$ contains explicitly the factor $j_p(kR_0)$ while the more general dependence on position has a $z_n(kR')$ of exactly the same cylinder function type as $z_\nu(kR)$.

Although this result appears to be different from the form in Eq. (A2-4), one expects on the basis of continuity of the two expansions across the surface $R' = R_0$ that they should be equivalent. That this is indeed so, and hence that Eq. (A2-4) truly represents the required coefficient for all R', is most easily shown by referring to the more general representations of $a(\mu, m \mid p, \nu, n)$ coefficients as "vector-coupling coefficients" or "Wigner $3-j$ symbols". The equivalence can then be shown to follow from symmetry properties of the Wigner $3-j$ symbols [20].

Again special recurrence relations are available [21], while others can be derived from Eq. (A2-2) by using well-known Legendre function recurrence formulas.

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SPIN MATRIX EXPONENTIALS AND TRANSMISSION MATRICES*

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Abstract. The three Pauli spin matrices $\sigma_i (i = 1, 2, 3)$ occur in the mechanical, especially quantum mechanical, theory of rotation in three-dimensional space. The three spin matrix exponentials are here defined as $\exp(\sigma_i x)$, where x is the independent variable. Transmission matrices can be expressed in terms of spin matrix exponentials, thereby permitting a more systematic treatment of transmission line circuits.

Introduction. In the design of quarter-wave transformers, it has hitherto always been assumed that the guide wavelength is independent of *position* along the line. This is so, for instance, for TEM modes; or for TE_{0n} modes in rectangular waveguide where the wide or 'a' dimension is kept constant. Such transformers, having guide wavelength independent of position, are called homogeneous transformers [1]. The first exact design formulas for ideal homogeneous quarter-wave transformers were given by Collin [2], who considered up to four sections. (The junction of two transmission lines when junction discontinuities are neglected, is called an "ideal transformer". This is analogous to two perfectly coupled coils of turns ratio $(Z_2/Z_1)^{1/2}$ and having infinite inductance.) The first complete synthesis procedure was given by Riblet [3]. The author later computed extensive numerical tables [4], which have been checked out experimentally on numerous occasions.

Riblet's synthesis procedure [3] is based on Richards' transformation [5] and Richards' theorem [6], and thereby depends on the commensurability of all transmission line sections in the circuit. The homogeneous quarter-wave transformer has also been used as a prototype circuit in the design of direct-coupled-cavity filters [7].

It has been shown that the performance of single-section quarter-wave transformers can always be improved by going from a homogeneous to an inhomogeneous design [8]. The analysis of inhomogeneous transformers of more than one section has only recently been undertaken [9], and the purpose of this paper is to present the mathematical tools which were developed for this purpose. A separate paper will deal with the design considerations and numerical results for multi-section inhomogeneous quarter-wave transformers [21].

Spin matrix exponentials. With line-lengths no longer commensurable, a more general formulation than is possible by Richards' transformation is required. For a systematic and compact treatment of transmission matrices, we shall employ the three Pauli spin matrices [10], which may be represented as follows:

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$$\left.\begin{aligned} \sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \sigma_2 &= j\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}\right\}. \tag{1}$$

They anti-commute among themselves, and their squares are equal to unity:

$$\left.\begin{aligned} \sigma_1\sigma_2 &= -\sigma_2\sigma_1 = j\sigma_3 \\ \sigma_2\sigma_3 &= -\sigma_3\sigma_2 = j\sigma_1 \\ \sigma_3\sigma_1 &= -\sigma_1\sigma_3 = j\sigma_2 \end{aligned}\right\} \tag{2}$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I, \tag{3}$$

where I is the unit matrix (idemfactor). We define the "spin matrix exponentials" by

$$E_i(x) = \exp(x\sigma_i) = I\cosh x + \sigma_i \sinh x \qquad (i = 1, 2, 3) \tag{4}$$

and their derivatives by

$$\left.\begin{aligned} E_i'(x) &= \frac{d}{dx}E_i(x) = \sigma_i \exp(x\sigma_i) \\ &= \sigma_i E_i(x) = I\sinh x + \sigma_i \cosh x \qquad (i = 1, 2, 3) \end{aligned}\right\}. \tag{5}$$

They do not commute among themselves unless i is the same, and then they commute and behave like ordinary exponentials:

$$E_i(x)E_i(y) = E_i(x + y). \tag{6}$$

Also,

$$E_i(x)\sigma_i = \sigma_i E_i(x) \tag{7}$$

but

$$E_i(x)\sigma_j = \sigma_j E_i(-x), \qquad i \neq j. \tag{8}$$

Similarly,

$$E_i'(x)E_i'(y) = E_i'(x + y), \tag{9}$$

$$E_i'(x)\sigma_i = \sigma_i E_i'(x), \tag{10}$$

$$E_i'(x)\sigma_j = -\sigma_j E_i'(-x), \qquad i \neq j. \tag{11}$$

Spin matrix exponentials occur in the mechanical (especially quantum mechanical) theory of rotation in three-dimensional space [11, 12]. The matrix product

$$Q = E_3\left(\frac{j\psi}{2}\right)E_1\left(\frac{j\theta}{2}\right)E_3\left(\frac{j\phi}{2}\right), \tag{12}$$

where θ, ϕ, ψ are the three Eulerian angles, yields a matrix whose four elements are the Cayley-Klein parameters [11]. At this point, we anticipate in order to complete the analogy: it will be seen later that Eq. (12) is like the transfer matrix of a single-section ideal transformer. This is not altogether surprising, since geometrical analogies have

been developed before [13, 14, 15], and projective charts have been used for the numerical solution of transmission line problems. The spinor theory of two-ports and its geometrical interpretation has also been discussed in general terms by Payne [16].

The Pauli spin matrices, together with the unit matrix, can be used to express any matrix [17] in the form of a quaternion, but the resulting more general form does not have the simplicity of the spin matrix exponentials which are well-suited for the analytic description and solution of inhomogeneous transformers.

The transfer matrix. There is no uniform terminology for the transformation matrices [18-20] which are used to analyze two-ports. The matrix defined by

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \mathbf{T} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}, \tag{13}$$

where a_1, b_1 are the incident and reflected wave amplitudes at the input, and a_2, b_2 those at the output (Fig. 1), will be referred to as the transfer matrix. As only lossless two-ports are here considered, wave amplitudes may be defined in terms of power by

$$\left.\begin{aligned} |a|^2 &= \text{power flow in the forward direction} \\ &\quad \text{(i.e. towards the load)} \\ |b|^2 &= \text{power flow in the backward direction} \\ &\quad \text{(i.e. towards the generator)} \end{aligned}\right\}. \tag{14}$$

The transmission coefficient, T, between two reference planes is the same (in both phase and amplitude) when going from left to right as when going from right to left. The transfer matrix can then be written [18]

$$\mathbf{T} = \begin{bmatrix} \dfrac{1}{T} & -\dfrac{\Gamma_2}{T} \\ \dfrac{\Gamma_1}{T} & T - \dfrac{\Gamma_1 \Gamma_2}{T} \end{bmatrix}, \tag{15}$$

where T is the (unique) transmission coefficient, Γ_1 is the reflection coefficient seen at the input (on the left) when a matched load is placed at the output (on the right), and Γ_2 is similarly defined for the output side.

We may, by appropriate choice of reference planes, let

$$\Gamma_1 = -\Gamma_2 = \Gamma \quad \text{(say)}. \tag{16}$$

From energy considerations:

1. Let $b_2 = 0$ in Fig. 1. Then

$$|\Gamma|^2 + |T|^2 = 1. \tag{17}$$

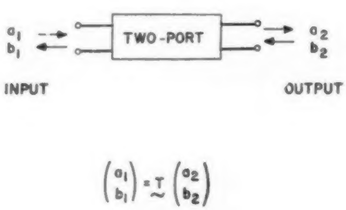

FIG. 1. Defining the transfer matrix.

2. Let $a_2 = 0$ in Fig. 1. Then

$$| T^2 + \Gamma^2 | = 1. \tag{18}$$

Comparing Eqs. (17) and (18), we infer that, under the condition (16), the transmission coefficient vector is parallel to the reflection coefficient vector [22]:

$$T \quad \text{parallel to} \quad \Gamma. \tag{19}$$

Now further choose the reference planes so that

$$\Gamma = \text{real}. \tag{20}$$

The transfer matrix then reduces to:

$$\mathbf{T} = \frac{1}{T}\begin{bmatrix} 1 & \Gamma \\ \Gamma & 1 \end{bmatrix}. \tag{21}$$

The diagonal and anti-diagonal parts of T. From Eq. (15) it follows that for a reflectionless two-port, **T** must be a diagonal matrix. Let

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{22}$$

Two useful concepts are

$$Di(\mathbf{T}) = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \tag{23}$$

$$= \text{diagonal part of } \mathbf{T}$$

$$Ag(\mathbf{T}) = \begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix} \tag{24}$$

$$= \text{anti-diagonal part of } \mathbf{T}.$$

For zero reflection,

$$Ag(\mathbf{T}) = 0. \tag{25}$$

The ideal transformer. The magnitude of the reflection coefficient at an ideal transformer (Fig. 2) is

$$| \Gamma | = \left| \frac{Z_2 - Z_1}{Z_2 + Z_1} \right|. \tag{26}$$

Select both reference planes to be coincident in the plane of the junction itself, and choose Γ to be real. This satisfies Eqs. (16) and (20) and therefore leads to the form (21) for the transfer matrix. Γ is now determined except for sign. In agreement with Ref. [4], we (arbitrarily) pick the positive sign and let

$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}. \tag{27}$$

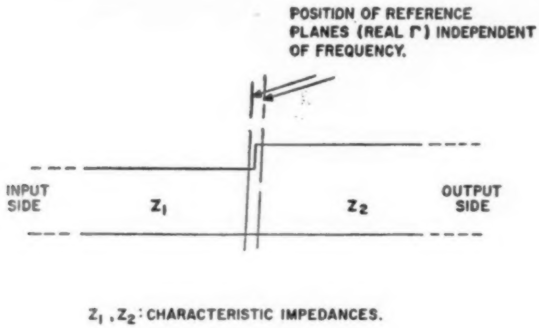

FIG. 2. Ideal transformer.

Defining the "junction *VSWR*" by

$$V = \frac{Z_2}{Z_1} \tag{28}$$

and the "log ratio" by

$$\alpha = \tfrac{1}{2} \ln V \tag{29}$$

the transfer matrix (21) of the ideal transformer reduces to

$$\mathbf{T} = E_1(\alpha) \tag{30}$$

which is the first spin matrix exponential as a function of the log ratio, α, of the junction.

$$\text{Also } \Gamma = \tanh \alpha, \tag{31}$$

and since Γ is real,

$$T = (1 - \Gamma^2)^{1/2} = \operatorname{sech} \alpha. \tag{32}$$

Length of transmission line. A section of transmission line of electrical length θ radians (Fig. 3) has a transfer matrix

$$\mathbf{T} = E_3(j\theta) \tag{33}$$

which is the third spin matrix exponential of imaginary argument $j\theta$.

The ABCD matrix. The ABCD matrix of an ideal transformer is

$$\mathbf{A} = E_3(\alpha) \tag{34}$$

and that of a length of transmission line is

$$\mathbf{A} = E_1(j\theta). \tag{35}$$

The transition between transfer and *ABCD* matrices has thus been effected merely by interchanging suffixes of the first and third spin matrix exponentials.

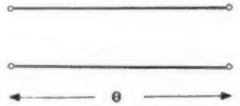

FIG. 3. Section of transmission line.

Conclusion. The spin matrix exponentials exp $(\sigma_i\, x)$, where x is either $j\theta(\theta =$ electrical line length) or the log ratio of a transformer, represent the transformation matrices of line sections and transformer steps. They are useful in the treatment of transmission line transformers, particularly inhomogeneous transformers, where the line lengths are incommensurable and Richards' transformation then does not apply [21].

Acknowledgment. The author owes much to the teaching and encouragement of Dr. W. H. Huggins of the Johns Hopkins University. The help of Dr. Ferdinand Hamburger, Jr. and Dr. C. F. Miller, also of the Johns Hopkins University, is gratefully acknowledged.

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22. The counterpart to Eq. (16) is $\Gamma_1 = \Gamma_2 = \Gamma$, which arises with symmetrical networks. In this case, the transmission coefficient and reflection coefficient vectors are orthogonal. See Leo Young, *A theorem on lossless symmetrical networks*, Trans. IRE **CT-7**, 75 (1960)

ON COMPLETE SOLUTIONS FOR FRICTIONLESS EXTRUSION IN PLANE STRAIN*

BY

J. M. ALEXANDER

Department of Mechanical Engineering, Imperial College

Consideration is given to the extension of partial slip line field upper bound solutions to give the true yield point load when the material is constrained. In particular, the problem of frictionless extrusion is studied and it is shown that it is possible to extend only one of three available partial solutions. If it is not possible to extend the available partial solution, the use of discontinuous stress fields leads to lower bound solutions, and examples of this technique are given.

1. Introduction. It is well known that all slip line field solutions constitute upper bound solutions, being partial incomplete solutions as described by Bishop (1953). This is because the slip line field is always associated with a kinematically admissible velocity field, although it is not generally associated with a statically admissible stress field, as discussed by Prager (1959, p. 118), for the extrusion problem. Bishop (1953) has proposed techniques for extending such partial stress solutions into neighbouring rigid regions, with a view to demonstrating whether or not a statically admissible stress field exists for the partial solution considered. If such a statically admissible stress field does exist, then the partial solution provides a lower bound also, giving in fact the true yield point load for the problem concerned. It is a consideration of the possibility of extending partial solutions to give true yield point loads for the problem of frictionless extrusion which forms the subject of this paper.

2. Available partial solutions, 3:1 ratio. For the 3:1thickness ratio of sheet extrusion through a square-edged die there exist three possible solutions, as shown in Fig. 1. All three solutions were proposed by Hill (1948) and in discussing the solutions shown in Fig. 1b and 1c he pointed out that the existence of dead metal regions would require the satisfying of certain conditions by the friction called into play on the surface of the dead metal.

Now the solution shown in Fig. 1(a) has the lowest yield point load, and since all three partial solutions are differing upper bound solutions, it should be possible to extend only that given by Fig. 1(a). This may not in fact be possible since there may be a further partial solution giving a still lower upper bound.

3. Complete solution, 3:1 ratio. A possible statically admissible stress field is shown in Fig. 2, for the partial solution of Fig. 1(a). Also shown in this figure is the stress plane showing the cycloidal traces of the pole of the Mohr's circle, using Prager's (1953) geometrical representation. The broken line $11 - 6'$ is a principal stress trajectory, and the stresses are transmitted across this boundary and supported on an infinite number of rectangular elements, via triangular elements of the type shown inset, the faces of the rectangular elements being normal and parallel to the extrusion axis. This concept was used by Bishop in the paper already referred to. The principal stress σ_p

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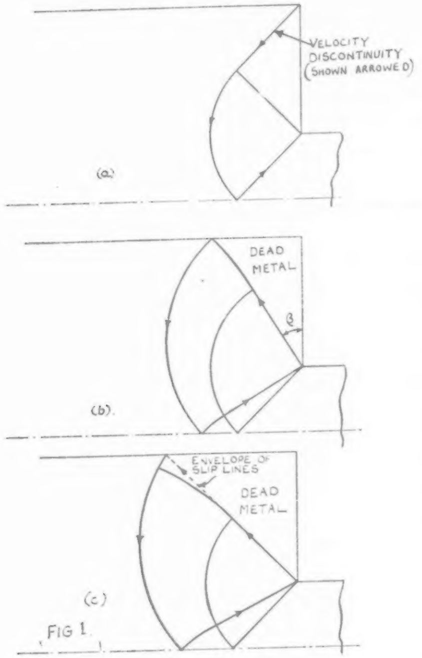
VELOCITY DISCONTINUITY (SHOWN ARROWED)
(a)
DEAD METAL
β
(b)
ENVELOPE OF SLIP LINES
DEAD METAL
(c)
FIG 1

Fig. 1

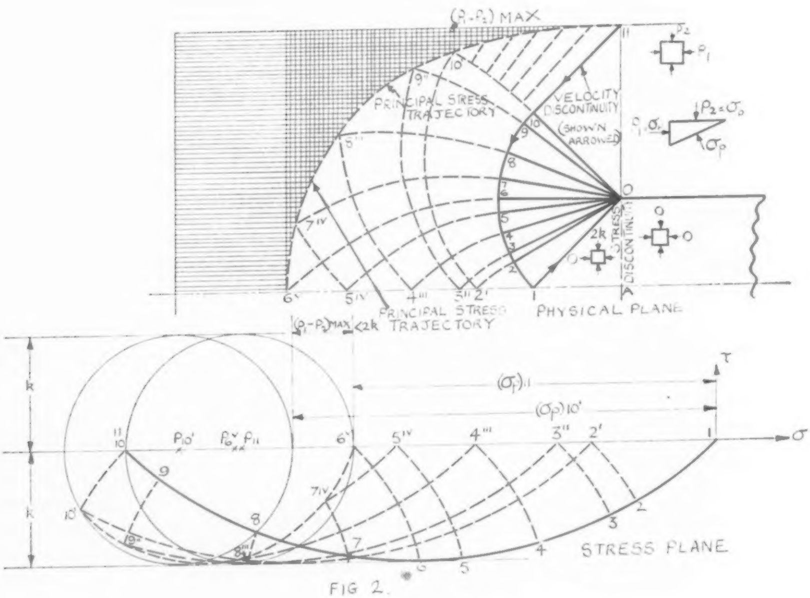
$(p_1 - p_2)$ MAX
PRINCIPAL STRESS TRAJECTORY
VELOCITY DISCONTINUITY (SHOWN ARROWED)
STRESS DISCONTINUITY
PRINCIPAL STRESS TRAJECTORY
PHYSICAL PLANE
$(p_1 - p_2)$MAX < 2k
$(\sigma_1)_{11}$
$(\sigma_p)_{10'}$
STRESS PLANE
FIG 2.

Fig. 2

on any triangular element can be supported by two direct stresses p_1 and p_2 both equal to σ_p, so that no shear stresses are invoked. The rigid material to the left of the principal stress trajectory will not yield provided that $| p_1 - p_2 | < 2k$. The maximum value of this stress difference occurs for the shaded element and is of amount $(\sigma_p)_{11} - (\sigma_p)_{10'}$ where $(\sigma_p)_{11}$ and $(\sigma_p)_{10'}$ are the principal stresses acting normal to the principal stress trajectory at the points 11 and 10′ respectively. The broken line $0A$ is a stress discontinuity, as suggested by Bishop, the stress states on each side of it being as indicated in the diagram. Since the infinity of columns transmit no shear stress, the boundary, equilibrium, and yield conditions are everywhere satisfied and the solution is a statically admissible stress field constituting a lower bound. Since it is also an upper bound, the actual yield point load will be associated with it.

Considering the solution shown in Fig. 1(b), it can be shown immediately that an associated statically admissible stress field does not exist, as follows. The angle β is less than $\pi/4$, and since the shear stress on the vertical die face must be made zero to comply with the requirements of the stress boundary conditions, the rigid dead metal vertex at 0 cannot support the boundary shear stresses postulated without yielding. That this is so can be seen from the analysis by Hill (1954) (case iv), in which it is proved that the angle β cannot be less than $\pi/4$.

Considering the solution shown in Fig. 1(c), an attempt to extend the partial solution is shown in Fig. 3. On the left hand side of the principal stress trajectory $10' - 6''$, the maximum stress difference (in the rectangular element A) is less than $2k$. On the right hand side of the envelope of slip lines $9 - 10'$, an equilibrium stress state on each triangular element can be as shown in the inset diagram, from which it is seen that

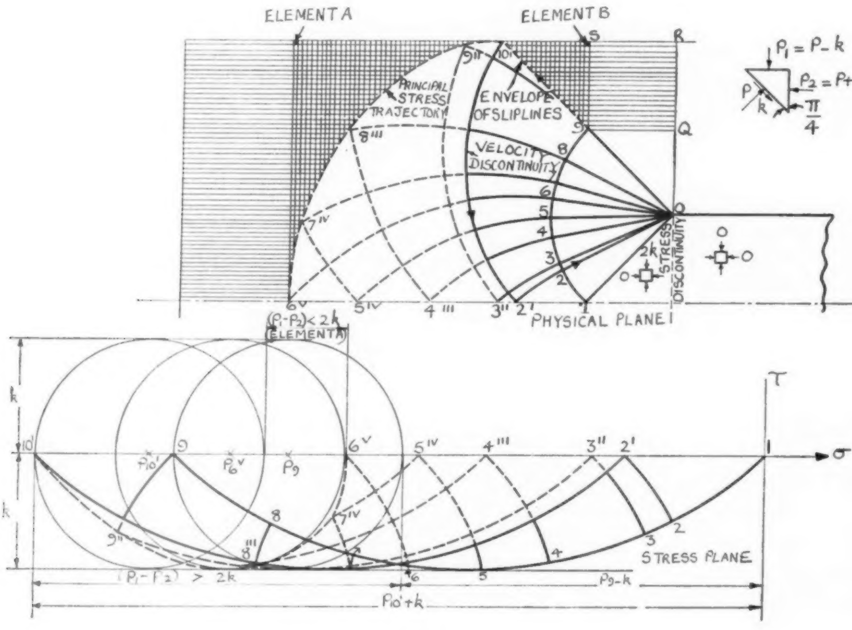

FIG. 3

$p_1 = p - k$, $p_2 = p + k$, where p is the mean stress at any point on the envelope. Thus at the point 9, $p_1 = p_9 - k$, and at the point 10′, $p_2 = p_{10'} + k$, as shown in the stress plane. Since the Mohr's circles for these two points are certainly not coincident, the element B is greatly overstressed in that $p_1 - p_2 \gg 2k$. This is true for the whole of the region $10' - R - Q - 9$ of the rigid dead metal, so that this region would yield under the distribution of stress applied to it. It is interesting to note that Hill's (1954) analysis is not applicable to the whole of the vertex $9 - 10' - S$, since the problem of the *variation* of stress on the face of such a vertex is not there considered. This extension of Bishop's infinity of rectangular elements permits the consideration of the possible yielding of such assumed rigid domains.

4. Extrusion ratios greater than 3:1. In Fig. 4 is shown the extended partial solution giving the true yield point load for extrusion ratios greater than 3:1, together with the stress plane, using the methods just described. The complete range of validity of this solution has not been investigated.

5. Other extrusion ratios. In Fig. 5 is shown a statically admissible stress field for an extrusion ratio of 2:1. This solution is interesting in that it has a dead metal zone, even with a frictionless interface between container and material. There is a field of constant stress in the dead metal region, the stress state being as shown in the inset element. It has not been found possible as yet to find complete solutions for other ratios.

6. Discussion. It has been demonstrated that the techniques suggested by Bishop

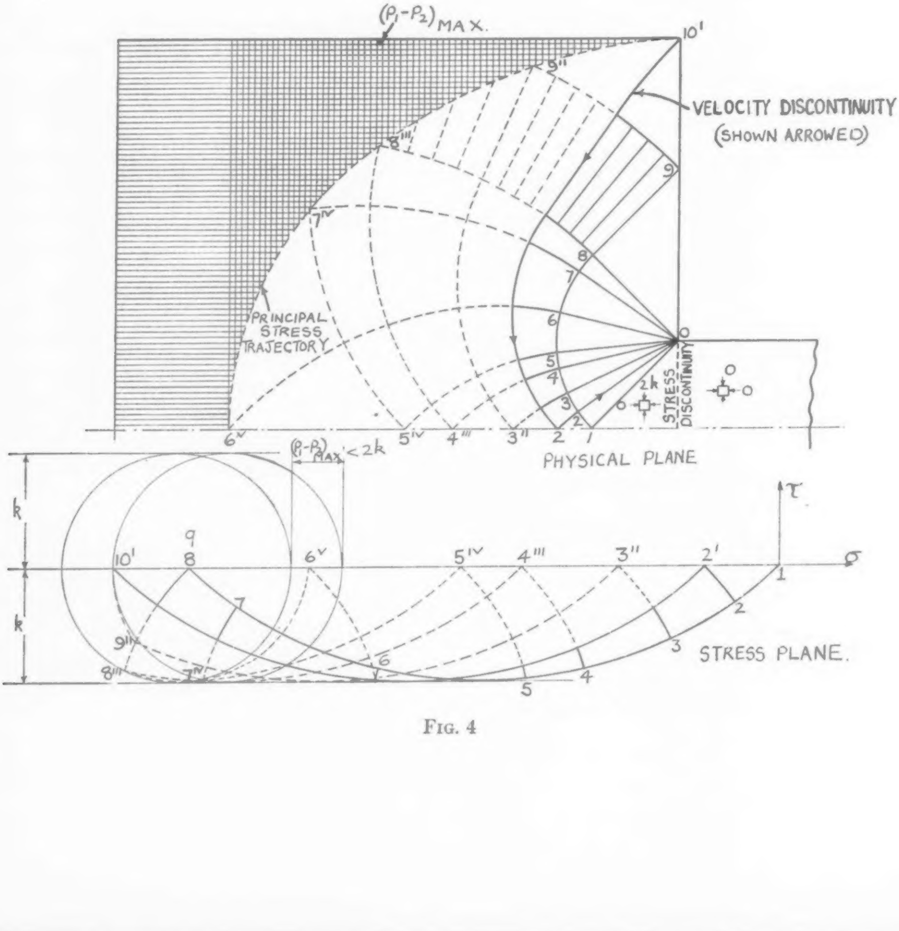

FIG. 4

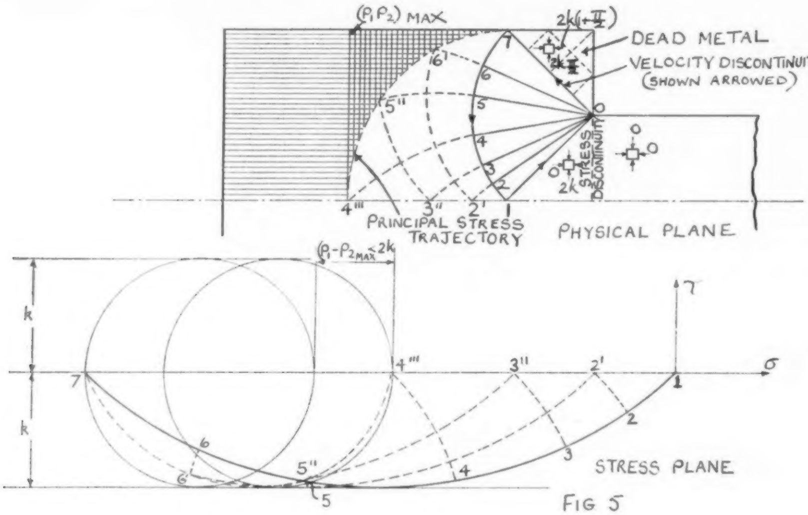

Fig. 5

for extending partial solutions can be applied to problems of contained deformation. In this connection, it is the author's opinion that Bishop's extension of Hill's field suggested in his paper (Bishop, 1953) is unacceptable because it violates the stress boundary conditions at the frictionless container—material interface. Problems of contained deformation with friction at such interfaces require special consideration, particularly concerning whether or not relative motion exists at such boundaries. The formulation of lower bound theorems for such cases constitutes a difficult problem.

It may often happen that *none* of the available partial solutions can be extended to give a statically admissible stress field, and hence the true solution. Under these circumstances a field of stress discontinuities in statical equilibrium may be found which would give a lower bound solution.

As examples, two discontinuous stress fields for the 3:1 ratio frictionless extrusion problem are shown in Figs. 6 and 7, in which the broken lines are the stress discontinuities separating regions of constant stress. The magnitudes of the constant stresses in each region are shown by the inset elements whose sides are parallel with the principal planes in the particular region concerned. In the stress plane of each diagram are shown the Mohr's circles for the states of stress in each of the regions, together with their 'poles', as defined by Prager (1953). The slip lines of both these fields intersect all boundaries and the axis of symmetry at angle $\pi/4$, so that they are statically admissible, and therefore lower bound solutions. The field shown in Fig. 6 leads to a mean extrusion pressure of $8/3k$, whilst that of Fig. 7 gives $10/3k$. (The field shown in Fig. 7 was derived from that suggested by Shield and Drucker (1953) for the indentation problem).

For this particular problem it has been possible to show that the upper bound slip line field of Fig. 2 gives, in fact, the true yield point load of the deformation, namely a mean extrusion pressure of $2k/3\ (2 + \pi) \simeq 10.284/3k$. Had this not been known, however, taking the mean extrusion pressure midway between the best available upper and lower bounds would have resulted in an error of about $1\frac{1}{2}\%$ only.

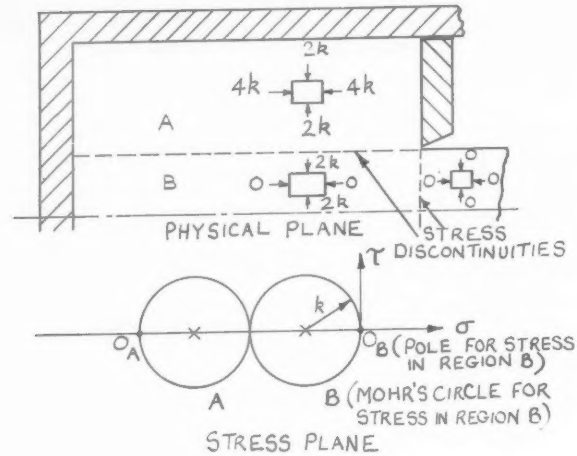
2k
4k
4k
A
2k
2k
B
0
0
2k
0
0
0
0
PHYSICAL PLANE
STRESS DISCONTINUITIES
τ
k
σ
O_A
O_B (POLE FOR STRESS IN REGION B)
A
B (MOHR'S CIRCLE FOR STRESS IN REGION B)
STRESS PLANE

Fig. 6

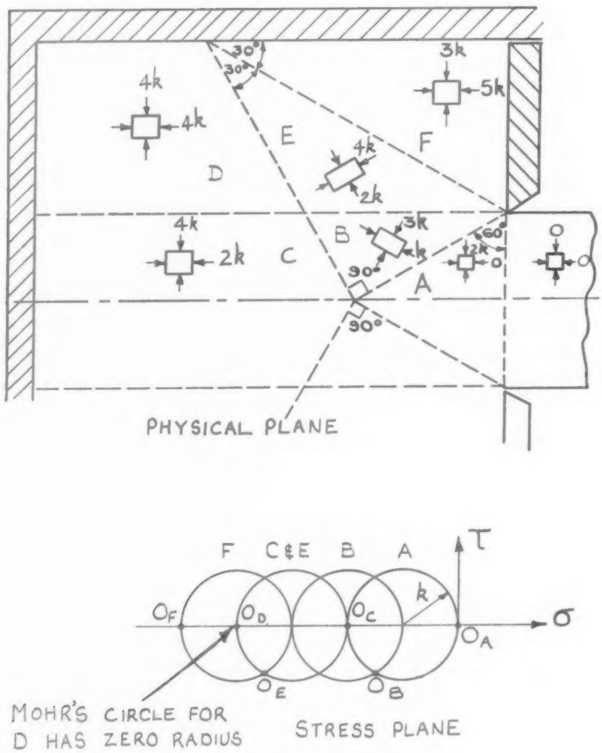
30°
30°
3k
5k
4k
4k
E
F
D
4k
2k
4k
2k
C
B
3k
k
60°
2k
0
0
0
90°
A
90°
PHYSICAL PLANE
F C&E B A
τ
k
O_F
O_D
O_C
O_A
σ
O_E
O_B
MOHR'S CIRCLE FOR D HAS ZERO RADIUS
STRESS PLANE

Fig. 7

Upper bound solutions for many problems of metal working have been proposed by Johnson (1959); the lower bound techniques proposed here form a complementary method of attacking such problems. The possibility of extending the upper bound methods to deal with three dimensional problems has been suggested previously, Alexander (1959); the use of discontinuous stress fields has been shown by Shield and Drucker (1953) to be suitable for application to three dimensional cases for finding lower bounds. Thus methods are available for bounding yield point loads for three-dimensional metal-working problems, which is a field requiring further study.

Acknowledgment. The author is indebted to Professor Hugh Ford, Professor of Applied Mechanics, Imperial College of Science and Technology, for many helpful and stimulating discussions relating to this work.

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BOOK REVIEWS

(Continued from p. 14)

The numerical treatment of differential equations. By L. Collatz. Third Edition. Springer-Verlag, Berlin, Gottingen, Heidelberg, 1960. xv + 568 pp. $23.56.

Though labelled "third edition", this is essentially an English translation of the second German edition (see this QUARTERLY, vol. **13**, p. 348). The references have been brought up to date, many improvements have been made, and the number of examples has been increased.

WILLIAM PRAGER

Operations research and systems engineering. Edited by C. D. Flagle, W. H. Huggins, and R. H. Ray. The Johns Hopkins Press, Baltimore, 1960. x + 889 pp. $14.50.

Part I (Perspectives) is concerned with the purpose and the historical development of operations research and systems engineering. Part II (Methodologies) is primarily devoted to the mathematical techniques. It contains chapters on simplified models (E. Naddor), basic statistics (A. J. Duncan), statistical quality control (A. J. Duncan), digital computers (W. C. Gore), inventory systems (E. Naddor), linear programming (V. V. McRae), queuing theory (C. D. Flagle), simulation techniques (C. D. Flagle), theory of games (E. Naddor), symbolic logic (W. E. Cushen), design of experiments (W. G. Cochran), human engineering (A. Chapanis), information theory (W. C. Gore), flow-graph representation of systems (W. H. Huggins), system dynamics (W. H. Huggins), and feedback and stability (N. H. Chosky). In Part III, the use of these principles and methods in operations research and systems engineering is illustrated by case studies.

W. PRAGER

Progress in solid mechanics. Volume I. Edited by I. N. Sneddon and R. Hill. North-Holland Publishing Co., Amsterdam, 1960. xii + 448 pp. $15.50.

This is the first volume of a new series, which is to be primarily devoted to the "basic principles and mathematical techniques of continuum mechanics, in all its aspects, together with experimental work of a fundamental kind". As the space available for this review is totally inadequate for a critical discussion of the eight articles, the following highly condensed table of contents will have to indicate the character of the volume.

Viscoelastic waves, by S. C. Hunter (Linear viscoelastic solid—Propagation of uniaxial stress pulses—Experimental investigations on pulse propagation—General equations of an isotropic viscoelastic solid).

Matrices of transmission in beam problems, by K. Marguerre (Vibrating beam—Matrices for points of discontinuity—Rigid supports,—matrix—"Stiff" supports—Application of matrix method to more complex problems).

Dynamic expansion of spherical cavities in metals, by H. G. Hopkins (Methods of approach and objectives—Hopkinson's size-scaling law—Strong discontinuities—Elastic deformations—Small elastic-plastic deformations—Large elastic-plastic deformations).

General theorems for elastic-plastic solids, by W. T. Koiter (Basic assumptions and stress-strain relations—Uniqueness theorems—Minimum principles—Plastic collapse theorems and limit analysis—Shakedown theorems—Existence of solutions).

Dispersion relations for elastic waves in bars, by W. A. Green (Exact solutions—Elementary approximate theory—Approximate methods—High frequency solutions for a circular cylinder).

Thermoelasticity, The dynamic theory, by P. Chadwick (Thermoelastic equations—Plane harmonic thermoelastic waves—Thermoelastic boundary value problems).

(Continued on p. 50)

ON THE THEORY OF THE PLASTIC POTENTIAL*

By HANS ZIEGLER, *Zürich*

1. **Introduction.** The theory of the plastic potential, proposed by v. Mises [1], connects the yield condition and the flow rule of a plastic solid. In its generalized form established by Prager [2], the theory may be stated as follows:

Let q_k ($k = 1, 2, \cdots, n$) be the generalized strains, i.e. a set of coordinates specifying the deformation of the plastic body in the sense of analytical mechanics, and let q_k^e, q_k^p denote respectively the elastic and the plastic components of q_k. If the work done on an infinitesimal increment of strain is given by

$$dA = Q_k\, dq_k\,, \tag{1.1}$$

where the summation convention has been applied, the Q_k are the generalized forces in the sense of analytical mechanics or, according to Prager, the generalized stresses corresponding to the coordinates q_k.

The generalized states of strain and stress may be depicted by points with position vectors $\mathbf{q}$, $\mathbf{Q}$ in Euclidean n-space R_n, or by the vectors $\mathbf{q}$, $\mathbf{Q}$ themselves. Defining the scalar product of two vectors in R_n by

$$\mathbf{Q}\cdot\mathbf{q} = Q_k q_k\,, \tag{1.2}$$

we obtain for the work (1.1) the representation

$$dA = \mathbf{Q}\cdot d\mathbf{q}. \tag{1.3}$$

Consider the solid in an arbitrary stage of the deformation process, and let $\mathbf{Q}$ denote the actual state of stress. Any state of stress $\mathbf{Q}^*$ which can be reached from $\mathbf{Q}$ without plastic flow will be called *nonplastic.* All of the $\mathbf{Q}^*$ constitute the nonplastic domain $\bar{R}_n$ in R_n, and any stress increment of the type $\mathbf{Q}^* - \mathbf{Q}$ will be denoted as nonplastic[1]). The yield limit is a hypersurface in R_n, defined by those nonplastic states of stress the infinitesimal changes of which are not exclusively nonplastic.

The theory of the plastic potential firstly stipulates that the elastic strains $\mathbf{q}^e$ follow from the stresses $\mathbf{Q}$ according to the laws of elasticity. The plastic strain increment $d\mathbf{q}^p$ corresponding to given values of the stress $\mathbf{Q}$ and the stress increment $d\mathbf{Q}$ is zero for any state of stress lying in the nonplastic domain but not at the yield limit. For states of stress at the yield limit, $d\mathbf{q}^p$ may be different from zero; it is secondly stipulated that

$$(\mathbf{Q}^* - \mathbf{Q})\cdot d\mathbf{q}^p \leq 0 \tag{1.4}$$

for any nonplastic stress increment $\mathbf{Q}^* - \mathbf{Q}$, and that

$$d\mathbf{Q}\cdot d\mathbf{q}^p \geq 0. \tag{1.5}$$

In virtue of (1.4) the nonplastic domain $\bar{R}_n$ is convex.

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[1]In Prager's or Koiter's terminology [3], $\bar{R}_n$ is the *elastic domain* and $\mathbf{Q}^*$ an *allowable state of stress.* The notations used here emphasize the fact that ideally plastic as well as hardening solids are considered.

2. Formulation of problem. So far, the theory of the plastic potential is but a hypothesis. This has been emphasized already by v. Mises; besides, his version was restricted to a volume element with strains ϵ_{ij} and stresses σ_{ij} . The generalization of Section 1 is due to Prager; the special form (1.4), (1.5) does not refer to a potential function and was given by Drucker [4].

Several authors have tried to provide a basis for the theory (see [3], p. 180). Bishop and Hill [5] have shown that (1.4) follows for the element of a polycrystalline aggregate from plausible assumptions concerning the behaviour of the single crystals. Drucker [4] has based the theory on a postulate regarding the work done on a prestressed element by an external agency in a cycle of application and removal. The author [6] has suggested a generalization of Onsager's theory of irreversible processes [7] to nonlinear cases, thus providing a thermodynamic basis for the theory of the plastic potential.

Either one of these approaches is based in its turn on certain postulates some of which are open to criticism [8]. Thus, it seemed worthwhile for once to reject any kind of postulate and to limit the scope of the investigation to a purely mathematical proof for the theory in Prager's generalized form, based on the sole assumption that it holds in v. Mises' sense for an element of volume. This proof has been provided by the author, in [8] for rigid-plastic materials and in [9] for elastic-plastic solids. The present paper gives a condensed version of the two articles and at the same time provides a simplification of the proof.

3. The rigid-plastic element. In a rigid-plastic solid, $\mathbf{q}^e = 0$ and hence $\mathbf{q} = \mathbf{q}^p$. The local states of strain and stress are given by ϵ_{ij} , σ_{ij} respectively and may be depicted by the vectors $\mathbf{e}$, $\mathbf{s}$ in Euclidean 9-space R_9 . Defining the scalar product in R_9 by

$$\mathbf{s}\cdot\mathbf{e} = \sigma_{ij}\epsilon_{ij} , \tag{3.1}$$

we obtain for the work per unit volume done on an infinitesimal increment of strain

$$d\bar{A} = \sigma_{ij}\, d\epsilon_{ij} = \mathbf{s}\cdot d\mathbf{e}. \tag{3.2}$$

In virtue of the symmetry of the strain and stress tensors, the vectors $\mathbf{e}$ and $\mathbf{s}$ actually lie in a linear subspace R_6 of R_9 . Here, the nonplastic domain may be defined according to the rules of Section 1. If we assume that the theory of the plastic potential is valid in v. Mises' sense, it follows from (1.4) and (1.5) that

$$(\mathbf{s}^* - \mathbf{s})\cdot d\mathbf{e} = (\sigma_{ij}^* - \sigma_{ij})\, d\epsilon_{ij} \leq 0 \tag{3.3}$$

for any nonplastic stress increment $\mathbf{s}^* - \mathbf{s}$, and that

$$d\mathbf{s}\cdot d\mathbf{e} = d\sigma_{ij}\, d\epsilon_{ij} \geq 0. \tag{3.4}$$

These relations also hold (with the equality sign) for states of stress below the yield limit. In virtue of (3.3) the nonplastic domain is convex.

4. The rigid-plastic solid. The states of strain and stress of the whole body B are given by the functions $\epsilon_{ij}(x_k)$, $\sigma_{ij}(x_k)$ respectively, either one of them depending on the coordinates x_k. These functions may be represented by vectors $\mathbf{E}$, $\mathbf{S}$ in function space F. Let the scalar product in F be defined by the volume integral

$$\mathbf{S}\cdot\mathbf{E} = \int_B \sigma_{ij}\epsilon_{ij}\, dV \tag{4.1}$$

extended over the entire body B. This definition is admissible (see [10]), since it satisfies the commutative law, $\mathbf{S}\cdot\mathbf{E} = \mathbf{E}\cdot\mathbf{S}$, the distributive law, $\mathbf{S}\cdot(\mathbf{E} + \mathbf{E}') = \mathbf{S}\cdot\mathbf{E} + \mathbf{S}\cdot\mathbf{E}'$, the

rule $\mathbf{S}\cdot\mathbf{O} = 0$ for multiplication by zero and the associative law for scalars, $(a\mathbf{S})\cdot\mathbf{E} = a(\mathbf{S}\cdot\mathbf{E})$. Besides, the definition provides F with a positive definite metric. On account of (4.1), the work done in an infinitesimal increment of strain is

$$dA = \int_B \sigma_{ij}\, d\epsilon_{ij}\, dV = \mathbf{S}\cdot d\mathbf{E}. \tag{4.2}$$

The representation considered here is not restricted to kinematically admissible states of strain nor to statically admissible states of stress; it holds for any states for which the integrals in (4.1), (4.2) exist. The nonplastic domain $\bar{F}$ in F may again be defined according to the rules of Section 1. In general, certain elements of B reach their local yield limit for states of stress $\mathbf{S}$ still inside the yield limit of the whole body. Plastic flow sets in when a sufficiently large domain of B has become plastic. At this stage, the state of stress in any element of B either lies below or on its local yield limit. Since none of the elements undergoes plastic flow in a nonplastic stress increment of the entire body, it follows from (3.3) that

$$(\mathbf{S}^* - \mathbf{S})\cdot d\mathbf{E} = \int_B (\sigma_{ij}^* - \sigma_{ij})\, d\epsilon_{ij}\, dV \leq 0 \tag{4.3}$$

for any nonplastic stress increment $\mathbf{S}^* - \mathbf{S}$. Also, on account of (3.4),

$$d\mathbf{S}\cdot d\mathbf{E} = \int_B d\sigma_{ij}\, d\epsilon_{ij}\, dV \geq 0. \tag{4.4}$$

Hence, the theory of the plastic potential, if valid for the element, likewise applies to the body as a whole. Incidentally, relations (4.3) and (4.4) also hold (with equality sign) for states of stress below the yield limit. In virtue of (4.3), also the nonplastic domain $\bar{F}$ is convex.

5. Generalized strains and stresses. Section **4** provides a basis for the rigorous treatment of a rigid-plastic body. In numerous cases, however, one is compelled to simplify the problem by introducing generalized strains and stresses as defined in Section 1. This means necessarily that only states of strain are considered which can be described by a finite set of parameters q_k $(k = 1, 2, \cdots, n)$. Such a reduction of the degree of freedom can be realized by introducing additional constraints; in certain cases also the elimination of originally existing constraints may result in a simplification (see examples in [8] and [9]). It is obvious that this process is only justified as long as it does not appreciably modify the actual state of strain.

The limitation considered here implies that only states of strain $\mathbf{E}$ are considered which belong to a certain subspace F_n of F. It does not involve a similar restriction with respect to the states of stress. Let $\mathbf{E}^{(k)}$ denote the state of strain in function space F corresponding to the generalized strain $q_k = 1$, $q_i = 0$ $(i \neq k)$. The subspace F_n then is defined by the states of strain

$$\mathbf{E} = q_k\mathbf{E}^{(k)}. \tag{5.1}$$

It is reasonable to assume that the vectors $\mathbf{E}^{(k)}$ are linearly independent. In this event, there is a one-to-one correspondence between the vectors $\mathbf{E}$ in F_n and $\mathbf{q}$ in R_n. On account of (4.2), (5.1) and (1.1), the work done by the stress $\mathbf{S}$ on an infinitesimal strain increment $d\mathbf{E}$ belonging to F_n is

$$dA = \mathbf{S}\cdot d\mathbf{E} = \mathbf{S}\cdot\mathbf{E}^{(k)}\, dq_k = Q_k\, dq_k\,. \tag{5.2}$$

Hence, the generalized stresses are given by the scalar products

$$Q_k = \mathbf{S}\cdot\mathbf{E}^{(k)}. \tag{5.3}$$

It follows immediately that any state of stress $\mathbf{S}$ in F is represented in R_n by a unique vector $\mathbf{Q}$ which, conversely, is the image of an infinity of vectors $\mathbf{S}$.

The straight section connecting the points in F with position vectors $\mathbf{S}'$, $\mathbf{S}''$ is given by

$$\mathbf{S} = \mathbf{S}' + a(\mathbf{S}'' - \mathbf{S}'), \qquad (0 \leq a \leq 1). \tag{5.4}$$

If $\mathbf{Q}$, $\mathbf{Q}'$, $\mathbf{Q}''$ are the vectors in R_n corresponding to $\mathbf{S}$, $\mathbf{S}'$, $\mathbf{S}''$, we obtain from (5.3) and (5.4)

$$Q_k = Q_k' + a(Q_k'' - Q_k'), \qquad (0 \leq a \leq 1). \tag{5.5}$$

Thus, the image in R_n of the section (5.4) is the straight section connecting the points with position vectors $\mathbf{Q}'$, $\mathbf{Q}''$.

Since the nonplastic domain $\bar{F}$ is convex, any vector $\mathbf{S}^* - \mathbf{S}$ representing a nonplastic stress increment lies entirely in $\bar{F}$. Its image in R_n is the vector $\mathbf{Q}^* - \mathbf{Q}$ which, according to the definition of the nonplastic domain, lies entirely in $\bar{R}_n$. Conversely, any vector $\mathbf{Q}^* - \mathbf{Q}$ in $\bar{R}_n$ is the image of at least one vector $\mathbf{S}^* - \mathbf{S}$ connecting two points of $\bar{F}$. Since $\bar{F}$ is convex, $\mathbf{S}^* - \mathbf{S}$ lies entirely in $\bar{F}$; hence, any vector in $\bar{R}_n$ can be considered the image of a nonplastic stress increment.

From (5.2) and (4.3) we obtain

$$(\mathbf{Q}^* - \mathbf{Q})\cdot d\mathbf{q} = (\mathbf{S}^* - \mathbf{S})\cdot d\mathbf{E} \leq 0, \tag{5.6}$$

where $\mathbf{Q}^* - \mathbf{Q}$ is an arbitrary nonplastic stress increment. Likewise, on account of (5.2) and (4.4),

$$d\mathbf{Q}\cdot d\mathbf{q} = d\mathbf{S}\cdot d\mathbf{E} \geq 0. \tag{5.7}$$

This is the proof that the theory of the plastic potential, if valid for the element, likewise applies to the treatment of the entire body in generalized strains and stresses. In virtue of (5.6), the nonplastic domain $\bar{R}_n$ is convex.

6. The elastic-plastic solid. In Section 1, no rules have been specified for the decomposition of the strain into its elastic and plastic components. In the case of a volume element, however, the decomposition is straightforward, provided the strains are sufficiently small[2]. Let us postulate that in the expression

$$d\bar{A} = \mathbf{s}\cdot d\mathbf{e}^e + \mathbf{s}\cdot d\mathbf{e}^p \tag{6.1}$$

following from (3.2) the first product represents the infinitesimal increase of elastic strain energy per unit volume, while the second one represents the work dissipated in the infinitesimal strain increment $d\mathbf{e}$. Then, $\mathbf{e}^e$ is the strain corresponding to the stress $\mathbf{s}$ according to the law of elasticity, and $\mathbf{e}^p$ is the permanent strain still present after removal of the stress.

In order to define a similar decomposition for the finite body, let us postulate that also in

$$dA = \mathbf{S}\cdot d\mathbf{E}^e + \mathbf{S}\cdot d\mathbf{E}^p \tag{6.2}$$

[2]For a few critical observations, see [9].

the products are respectively the increment of elastic strain energy and the work of dissipation. Thus, $\mathbf{E}^e$ is built up from the local elastic strains $\mathbf{e}^e$ and $\mathbf{E}^p$ from the local plastic strains $\mathbf{e}^p$. Here, $\mathbf{E}^e$ may be different from the state of strain corresponding to the given loads if the body were elastic; also, $\mathbf{E}^p$ is not necessarily the state of strain after removal of the loads. On the other hand, the definition (6.2) implies that the work of dissipation is

$$d\bar{A}^p = \mathbf{s}\cdot d\mathbf{e}^p = \sigma_{ij}\, d\epsilon_{ij}^p \tag{6.3}$$

per unit volume and

$$dA^p = \mathbf{S}\cdot d\mathbf{E} = \int_B \sigma_{ij}\, d\epsilon_{ij}^p\, dV \tag{6.4}$$

for the entire body.

Let us now replace (3.2) by (6.3) and (4.2) by (6.4). Assuming that relations (3.3) and (3.4) hold for the plastic strain increment $d\mathbf{e}^p$ instead of $d\mathbf{e}$, and retracing the demonstration of Section 4, we arrive at (4.3) and (4.4) with ($d\epsilon_{ij}^p$ instead of $d\epsilon_{ij}$ and) $d\mathbf{E}^p$ instead of $d\mathbf{E}$.

For the representation in generalized coordinates, the situation is similar. Let us postulate that the products in

$$dA = \mathbf{Q}\cdot d\mathbf{q}^e + \mathbf{Q}\cdot d\mathbf{q}^p \tag{6.5}$$

are respectively equal to the increment of elastic strain energy and the work of dissipation. Then (5.2) can be replaced by

$$dA^p = \mathbf{S}\cdot d\mathbf{E}^p = \mathbf{Q}\cdot d\mathbf{q}^p, \tag{6.6}$$

and the remainder of Section **5** leads to (5.6) and (5.7) with ($d\mathbf{E}^p$ instead of $d\mathbf{E}$ and) $d\mathbf{q}^p$ instead of $d\mathbf{q}$.

Thus, the results proved in Sections **4** and **5** also hold for the plastic strains in elastic-plastic solids, provided the elastic and plastic strain components are defined by means of strain energy and dissipation work[3].

7. Observations. In an elastic-plastic solid, the elements which reach their local yield limit are not surrounded by rigid material. Hence, plastic flow sets in as soon as the first elements become plastic. It follows that under otherwise identical circumstances the nonplastic domain of the elastic-plastic solid is usually distinct from the one of the rigid-plastic body. Moreover, any plastic flow is apt to modify the nonplastic domain. Hence, the yield limit of the elastic-plastic solid undergoes a continuous transformation as the plastic flow proceeds towards collapse. On the other hand, the yield limit of a non-hardening rigid-plastic solid is always the same.

In order to determine the nonplastic domain $\bar{R}_n$ in an arbitrary stage of the deformation process, it is necessary to consider the actual state of stress $\mathbf{S}$. From the generalized stresses Q alone no information concerning the yield limit is available. In the case of a non-hardening rigid-plastic body, the yield limit must be determined only once. Once it is known, the theory of the plastic potential may be applied in the sense of Section **1**, and the problem can be treated henceforth in generalized strains and stresses. In fact, this is the reason why Prager's version of the theory offers an essential simpli-

[3]As, e.g., in [6].

fication. In the case of an elastic-plastic solid, however, one is compelled to keep track of the continuous changes in shape of the yield limit. Since this requires that the actual state of stress **S** be pursued throughout the deformation process, the use of generalized strains and stresses does not seem to offer any advantages here, except, of course, in limit analysis.

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MOMENTS OF THE GENERALIZED RAYLEIGH DISTRIBUTION*

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I. Introduction. Gaussian processes are of considerable interest in problems involving random noise. Also of interest is the Rayleigh distribution which arises in work on radar, the detection of signals in noise, etc. [1, 2]. The generalized Rayleigh process promises to be of interest in the future especially when signals in noise are thought of to exist in a finite dimensional Hilbert space [3, 4, 5]. The generalized Rayleigh process was defined and some of its properties were investigated by K. S. Miller, *et al.* [6]. The purpose of this paper is to investigate the moments of the generalized Rayleigh distribution.

Let X_1, X_2, $\cdots$ X_N be independent Gaussian random variables with means $\bar{x}_1$, $\bar{x}_2$, $\cdots$ $\bar{x}_N$ respectively and equal variances of one. A generalized Rayleigh random variable, Y, (also referred to as a non-central chi-square variable) is defined as

$$Y^2 = \sum_{i=1}^{N} X_i^2 \tag{1.1}$$

and the density function of Y, denoted $g(y)$, is given by [Ref. 6, Eq. 1.6]

$$g(y) = \begin{cases} y_0(y/y_0)^{N/2} \exp\left[(y_0^2 + y^2)/2\right] I_{(N-2)/2}(y_0 y) & \text{for} \quad y > 0 \\ 0 & \text{for} \quad y \leqq 0, \end{cases} \tag{1.2}$$

where

$$y_0^2 = \sum_{i=1}^{N} \bar{x}_i^2 \tag{1.3}$$

and $I_k(x)$ is the modified Bessel function of the first kind. $g(y)$ is called the generalized Rayleigh distribution. In this paper expressions for the moments about zero of $g(y)$ and several interesting properties of these moments will be derived.

It does not complicate the problem to consider non-integer moments. Therefore, the ath moment of $g(y)$ is given by

$$M_a(N, y_0) = \int_{-\infty}^{\infty} y^a g(y)\, dy, \tag{1.4}$$

where "a" is any real number. (However, as will be seen later, the above integral exists only for $a > -N$ hence a can be any real number greater than $-N$.) Whenever only integral moments are considered the subscript n will be used.

The important results are:

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(1) The power series expression for $M_a(N, y_0)$,

$$M_a(N, y_0) = 2^{a/2} \exp(-y_0^2/2) \sum_{r=0}^{\infty} \frac{\Gamma[r + (N + a)/2]}{r!\, \Gamma[r + (N/2)]} (y_0^2/2)^r, \tag{1.5}$$

(2) The "closed form" expression for $M_a\ (N, y_0)$,

$$M_a(N, y_0) = 2^{a/2} \exp(-y_0^2/2) \frac{\Gamma[(N + a)/2]}{\Gamma(N/2)} M[(N + a)/2, N/2, y_0^2/2], \tag{1.6}$$

where M (with no subscript) is the confluent hypergeometric function (in the notation of Jahnke and Ende, [7, p. 275]).

(3) The asymptotic expressions for $M_a\ (N, y_0)$,

$$M_a(N, y_0) \sim y_0^a\left[1 + \frac{a(N + a - 2)}{2y_0^2} + \frac{a(a - 2)(N + a - 2)(N + a - 4)}{2!\,(2y_0^2)^2} + \cdots\right] \quad \text{as} \quad y_0 \to \infty, \tag{1.7}$$

and

$$M_a(N, y_0) \sim N^{a/2} \exp(ay_0^2/2N) \quad \text{as} \quad N \to \infty. \tag{1.8}$$

(4) The recursion formulas

$$M_{a+2}(N, y_0) = (N + a)M_a(N, y_0) + y_0^2\left[M_a(N, y_0) + y_0 \frac{dM_a(N, y_0)}{dy_0}\right] \tag{1.9}$$

and

$$M_{a-2}(N, y_0) = \tfrac{1}{2} \exp(-y_0^2/2)(y_0^2/2)^{(2-N-a)/2} \int_0^{y_0^2/2} \exp(x) x^{(N+a-4)/2} M_a[(2x)^{1/2}]\, dx \quad \text{for} \quad a > 2 - N. \tag{1.10}$$

(5) The upper bounds on negative integer moments

$$M_{-n} \leqq |\, 1/y_0 \,|^n \qquad n = 1, 2, \cdots N - 2 \tag{1.11}$$

and

$$M_{-n} \leqq |\, 1/(N - n) \,|^{n/2} \qquad n = 1, 2, \cdots N. \tag{1.12}$$

II. General expressions for the moments of the generalized Rayleigh distribution. We first obtain a simple expression for the moments as defined by Eq. (1.4). Substituting (1.2) in (1.4) and abbreviating $M_a\ (N, y_0)$ by M_a we obtain

$$M_a = y_0^{1-(N/2)} \exp(-y_0^2/2) \int_0^{\infty} y^{(2a+N)/2} \exp(-y^2/2) I_{(N-2)/2}(yy_0)\, dy. \tag{2.1}$$

The integral in the above expression diverges if $a \leqq -N$. Therefore, whenever a is used it will denote a real number greater than $-N$ and n will denote an integer greater than $-N$. Substituting in (2.1) the equivalent power series for $I_{(N+2)/2}\ (yy_0)$ [8, p. 163], we obtain

$$M_a = 2^{1-(N/2)} \exp(-y_0^2/2) \int_0^{\infty} \sum_{r=0}^{\infty} \frac{y^{N+2r+a-1} \exp(-y^2/2)}{r!\, \Gamma[(N + 2r)/2]} (y_0/2)^{2r}\, dy. \tag{2.2}$$

Interchanging summation and integration in the above, we note that the integral is a gamma function, hence Eq. (1.5).

The sum in (1.5) can, recalling that $B\Gamma(B) = \Gamma(B + 1)$, be rewritten,

$$\sum_{r=0}^{\infty} \frac{\Gamma[r + (N + a)/2]}{r!\,\Gamma[r + (N/2)]} (y_0^2/2)^r = \frac{\Gamma[(N + a)/2]}{\Gamma(N/2)} \cdot \left\{1 + \sum_{r=1}^{\infty} \frac{1}{r!} \left[\prod_{k=0}^{r-1} (N + a + 2k)/(N + 2k)\right] (y_0^2/2)^r\right\}. \tag{2.3}$$

The term in brackets in (2.3) is the power series for the confluent hypergeometric function, $M[(N + a)/2, N/2, y_0^2/2]$, and will sometimes be abbreviated by M. Therefore, using (2.3) in (1.5) we obtain the closed form expression (1.6).

III. Asymptotic expressions. In this section asymptotic expressions for $M_a(N, y_0)$ as a function of y_0 with a and N fixed and as a function of N with a and y_0 fixed will be derived.

First consider the case when a and N are fixed. $M_a(N, y_0)$ is given by (1.6) in terms of the hypergeometric function and therefore, using the asymptotic expression for M given on p. 275 of [7], we obtain (1.7), the asymptotic expression for $M_a(N, y_0)$ as $y_0 \to \infty$.

To obtain the asymptotic expression for M_a as $N \to \infty$ we use the power series expression for M_a given in (1.5) where the series has been rewritten as shown in (2.3). The product in the right hand side of (2.3) can be written

$$\prod_{k=0}^{r-1} (N + a + 2k)/(N + 2k) = [1 + (a/N)]^r \prod_{k=0}^{r-1} \frac{(N + 2k + a)N}{(N + a)(N + 2k)} = [1 + (a/N)]^r \prod_{k=0}^{r-1} \left[1 - \frac{2ka}{(N + a)(N + 2k)}\right]. \tag{3.1}$$

It is clear from (3.1) that this product is asymptotic to $[1 + (a/N)]^r$. Hence from (2.3) and (1.5) we obtain

$$M_a(N, y_0) \sim \frac{\Gamma[(N + a)/2]}{\Gamma(N/2)} 2^{a/2} \exp(ay_0^2/2N) \quad \text{as} \quad N \to \infty. \tag{3.2}$$

But

$$\frac{\Gamma[(N + a)/2]}{\Gamma(N/2)} \sim (N/2)^{a/2} \quad \text{as} \quad N \to \infty \tag{3.3}$$

and hence Eq. (1.8).

IV. Recursion formulas. Recursion formulas can be easily derived for moments of order $a + 2$ in terms of the moment of order a and its derivative with respect to y_0. Since $M_0(N, y_0) = 1$, moments of even integer order are easy to compute from this formula. A recursion formula for moments of order $a - 2$ is also obtained in terms of an integral involving the moment of order a.

As a preliminary step, consider Eq. (1.5) where x has been used in place of $y_0^2/2$. After differentiating with respect to x and rearranging terms we obtain

$$2x\left(M_a + \frac{dM_a}{dx}\right) = 2 \cdot 2^{a/2} \exp(-x) \sum_{r=0}^{\infty} \frac{rx^r}{r!} \frac{\Gamma[r + (N + a)/2]}{\Gamma[r + (N/2)]}. \tag{4.1}$$

But

$$M_{a+2} = 2 \cdot 2^{a/2} \exp(-x) \sum_{r=0}^{\infty} \frac{x^r}{r!} \frac{[r + (N + a)/2]\Gamma[r + (N + a)/2]}{\Gamma[r + (N/2)]}. \tag{4.2}$$

Therefore

$$M_{a+2} = (N + a)M_a + 2x\left(M_a + \frac{dM_a}{dx}\right). \tag{4.3}$$

If we put (4.3) in terms of y_0 we obtain the recursion formula given in (1.9). In particular since $M_0 = 1$

$$\begin{aligned} M_2 &= N + y_0^2, \\ M_4 &= N(N + 2) + 2y_0^2(N + 2) + y_0^4, \\ &\text{etc.} \end{aligned} \tag{4.4}$$

To obtain the recursion formulas for decreasing moments we multiply $M_a(x)$ by $\exp(x)x^{(N+a-4)/2}$ and integrate term by term from 0 to $y_0^2/2$ resulting in

$$\int_0^{y_0^2/2} \exp(x)x^{(N+a-4)/2}M_a[(2x)^{1/2}]\,dx$$
$$= (y_0^2/2)^{(N+a-2)/2}2^{a/2} \sum_{r=0}^{\infty} \frac{\Gamma[r - 1 + (N + a)/2]}{r!\,\Gamma[r + (N/2)]} (y_0^2/2)^r. \tag{4.5}$$

But

$$M_{a-2} = 2^{(a-2)/2} \exp(-y_0^2/2) \sum_{r=0}^{\infty} \frac{\Gamma[r - 1 + (N + a)/2]}{r!\,\Gamma[r + (N/2)]} (y_0^2/2)^r \tag{4.6}$$

which together with (4.5) gives Eq. (1.10). It can be shown that the integration in (4.5) exists only if $a > 2 - N$.

When $a = 0$ (1.10) becomes

$$M_{-2}(N, y_0) = \tfrac{1}{2} \exp(-y_0^2/2)(y_0^2/2)^{(2-N)/2} \int_0^{y_0^2/2} \exp(x)x^{(N-4)/2}\,dx \tag{4.7}$$

and for a few particular values of N we have

$$M_{-2}(4, y_0) = 1/y_0^2,$$
$$M_{-2}(6, y_0) = [1 - (2/y_0^2)]/y_0^2$$

and

$$M_{-2}(8, y_0) = [1 - (4/y_0^2) + (8/y_0^4)]/y_0^2. \tag{4.8}$$

V. Upper bounds on negative integer moments. Two very simple expressions which are upper bounds on negative integer moments can easily be derived. Let n be a negative even integer, say $n = -2m$, $m = 1, 2, \cdots$ then 1.5 can be written

$$M_{-2m} = 2^{-m} \exp(-y_0^2/2) \sum_{r=0}^{\infty} \frac{1}{r!} (y_0^2/2)^r \prod_{k=1}^{m} 2/(N + 2r - 2k). \tag{5.1}$$

Assuming $(N/2) - m > 0$ the right side of (5.1) is not decreased by letting r be zero in each term of the product resulting in

$$M_{-2m} \leqq 1/(N - 2m)^m. \tag{5.2}$$

Alternately if we assume $(N/2) - 1 \geqq m$ we can replace m for $(N/2) - 1$ in the product in Eq. (5.1) and not decrease the right hand side. Therefore we have

$$M_{-2m} \leqq 2^{-m} \exp(-y_0^2/2) \sum_{r=0}^{\infty} \frac{1}{(r+m)!} (y_0^2/2)^r. \tag{5.3}$$

The sum in the right hand side of (5.3) can be written

$$\sum_{r=0}^{\infty} \frac{1}{r!} (y_0^2/2)^r \frac{r!}{(r+m)!} = (2/y_0^2)^m \sum_{r=m}^{\infty} \frac{1}{r!} (y_0^2/2)^r. \tag{5.4}$$

By filling in some positive terms in the sum on the right hand side of (5.4) it becomes an exponential and we obtain

$$M_{-2m} \leqq 1/y_0^{2m}. \tag{5.5}$$

By Schwartz's inequality

$$M_{m+k}^2 \leqq M_{2m} M_{2k}. \tag{5.6}$$

In particular if $m = 0$ and k is a negative integer, i.e., $k = -n$, $n = 1, 2, \cdots$ then

$$M_{-n}^2 \leqq M_{-2n}. \tag{5.7}$$

Combining (5.7) with (5.5) and (5.2) we obtain (1.11) and (1.12) respectively.

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BOOK REVIEWS

(*Continued from p. 38*)

Continuous distributions of dislocations, by B. A. Bilby (Burgers vector and torsion tensor—Shape, lattice and dislocation deformations—Zero lattice pure strain—Dislocation density and stress—Generalized spaces).

Asymmetric problems of the theory of elasticity for a semi-infinite solid and a thick plate, by R. Muki (Solution of the equations of equilibrium by Hankel transforms—Solution of the thermo-elastic equations by Hankel transforms—Stresses in a semi-infinite elastic solid under the compressive action of a rigid body—Stresses in a semi-infinite elastic solid with a tangential load on its surface—Thermal stresses in a semi-infinite elastic solid and a thick plate under steady distribution of temperature).

W. Prager

Dynamic programming and Markov processes. By Ronald A. Howard. Technology Press of M. I. T., and John Wiley & Sons, New York and London, 1960. viii + 136 pp. $5.75.

Consider a physical system S represented at any time t by a state vector $x(t)$. The classical description of the unfolding of the system overtime uses an equation of the form $x(t) = F(x(s), s \leq t)$, where F is a prescribed operation upon the function $x(s)$ for $s \leq t$. In certain simple cases, this reduces to the usual vector differential equation $dx/dt = g(x)$, $x(0) = c$.

For a variety of reasons, it is sometimes preferable to renounce a deterministic description and to introduce stochastic variables. If we take $x(t)$ to be a vector whose i-th component is now the probability that the system is in state i at time t, and allow only discrete values of time, we can in many cases describe the behavior of the system over time quite simply by means of the equation $x(t + 1) = Ax(t)$. Here $A = (a_{ij})$, $i, j = 1, 2, \cdots, N$, is a transition matrix whose element a_{ij} is the probability that a system in state j at time t will be found in state i at time $t + 1$. Processes of this type are called Markov processes and are fundamental in modern mathematical physics.

So far we have assumed that the observer plays no role in the process. Let us now assume that in some fashion or other the observer has the power to choose the transition matrix A at each stage of the process. We call a process of this type a *Markovian decision process*. It is a special, and quite important, type of dynamic programming processes; cf. Chapter XI of *Dynamic Programming*, Princeton University Press, 1957.

Let us suppose that at any stage of the process, we have a choice of one of a set of matrices, $A(q) = (a_{ij}(q))$. Associated with each choice of q and initial state i is an expected single-stage return $b_i(q)$. We wish to determine a sequence of choices which will maximize the expected return from n stages of the process. Denoting the maximum expected return from an n-stage process by $f_i(n)$, the principle of optimality yields the functional equation

$$f_i(n) = \max_q \left[b_i(q) + \sum_{j=1}^{N} a_{ij}(q) f_j(n - 1) \right].$$

In this form, the determination of optimal policies and the maximum returns is easily accomplished by means of digital computers; cf. S. Dreyfus, *J. Oper. Soc. of Great Britian*, 1958. Problems leading to similar equations, resolved in similar fashion, arise in the study of equipment replacement and in continuous form in the "optimal inventory" problem; see Chapter Five of the book mentioned above and K. D. Arrow, S. Karlin, and H. Scarf, *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, 1959.

As in the case of the ordinary Markov process, a question of great significance is that of determining the asymptotic behavior as $n \to \infty$. It is reasonable to suspect, from the nature of the underlying decision process, that a certain steady-state behavior exists as $n \to \infty$. This can be established in a number of cases.

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HIGH FREQUENCY VIBRATIONS OF CRYSTAL PLATES*

BY

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1. Introduction. In this paper, Cauchy's [1] two-dimensional equations of coupled flexural and extensional motion of crystal plates are extended to the next higher order of approximation so as to accommodate the frequencies of the two lowest thickness-shear modes. The equations obtained are also extensions of previous equations [2] in which flexure and the same thickness-shear modes were included, but coupling with extensional modes was omitted. The new equations are deduced from the three-dimensional equations of linear elasticity by a procedure based on the series expansion methods of Cauchy [1] and Poisson [3] and the variational method of Kirchhoff [4]. Comparison of the appropriate solution of the resulting equations with Ekstein's [5] solution of the three-dimensional equations, for an infinite plate, reveals close agreement between the two frequency spectra, over the extended range of frequencies, for all five branches of the spectrum of the two-dimensional equations. This indicates that solutions of these equations, for bounded plates, will give reliable results, over the extended range of frequencies, since the modes of bounded plates, in that range, are composed essentially of coupled overtones of the first five modes of vibration of an infinite plate.

In addition to the derivation of the approximate equations, theorems of uniqueness and orthogonality are established and some general conclusions are drawn regarding solutions in rectangular coordinates and vibrations of rectangular plates.

2. Expansion in power series. The plate is referred to rectangular coordinates x_i $(i = 1, 2, 3)$ with x_1 and x_3 in the middle plane and the faces at $x_2 = \pm h$. The components of displacement u_j $(j = 1, 2, 3)$ are expanded in power series of the thickness-coordinate:

$$u_j = \sum_{n=0}^{\infty} x_2^n u_j^{(n)} = u_j^{(0)} + x_2 u_j^{(1)} + x_2^2 u_j^{(2)} \cdots , \tag{1}$$

where the $u_j^{(n)}$ are functions of x_1, x_3 and the time, t, only.

Stress-equations of motion. The series expression for u_j is substituted in

$$\int_V (T_{ij,i} - \rho u_{j,tt})\, \delta u_j \, dV = 0 \tag{2}$$

which is obtained from the variational equation of motion, deduced from Hamilton's principle [6]. In (2), the integration is over the volume, V, of the plate; the T_{ij} are the components of stress; ρ is the density; and the summation convention for repeated indices is employed, as is the comma notation for differentiation with respect to the x_i and the time t.

When the integration with respect to x_2, from $-h$ to h, is performed in (2), the result is

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$$\int_A \sum_{n=0}^{\infty} \left(T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} - \rho \sum_{m=0}^{\infty} H_{mn} u_{j,tt}^{(m)} \right) \delta u_j^{(n)} \, dA = 0, \tag{3}$$

where A is the area of the plate and

$$T_{ij}^{(n)} \equiv \int_{-h}^{h} x_2^n T_{ij} \, dx_2 , \qquad F_j^{(n)} \equiv [x_2^n T_{2j}]_{-h}^{h}$$

$$H_{mn} \equiv \begin{cases} 2h^{m+n+1}/(m+n+1), & m+n \quad \text{even} \\ 0, & m+n \quad \text{odd} \end{cases}. \tag{4}$$

Since (3) must hold for all A and arbitrary $\delta u_j^{(n)}$, the quantity in parentheses must vanish and we arrive at the stress-equations of motion of order n

$$T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} = \rho \sum_{m=0}^{\infty} H_{mn} u_{j,tt}^{(m)} .$$

Strain. The three-dimensional components of strain, S_{ij}, are expressed in terms of the u_i by

$$S_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{5}$$

Upon substituting (1) in (5) we obtain, after a rearrangement of terms,

$$S_{ij} = \sum_{n=0}^{\infty} x_2^n S_{ij}^{(n)}, \tag{6}$$

where

$$S_{ij}^{(n)} \equiv \tfrac{1}{2}[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)(\delta_{2j} u_i^{(n+1)} + \delta_{i2} u_j^{(n+1)})]$$

in which δ_{ij} is the Kronecker symbol.

Stress-strain relations. In three dimensions,

$$T_{ij} = c_{ijkl} S_{kl} , \qquad c_{ijkl} = c_{jikl} = c_{klij} , \tag{7}$$

$$S_{ij} = s_{ijkl} T_{kl} , \qquad s_{ijkl} = s_{jikl} = s_{klij} , \tag{8}$$

where c_{ijkl} and s_{ijkl} are the elastic stiffnesses and compliances, respectively.

The expressions for the two-dimensional $T_{ij}^{(n)}$ in terms of the $S_{ij}^{(n)}$ are obtained by inserting (6) in (7) and (7) in the first of (4), with the result

$$T_{ij}^{(n)} = c_{ijkl} \sum_{m=0}^{\infty} H_{mn} S_{kl}^{(m)}. \tag{9}$$

Energy-densities. A strain-energy-density, U, and a kinetic energy-density, K, both reckoned per unit area of the plate, are defined by

$$U \equiv \tfrac{1}{2} \int_{-h}^{h} c_{ijkl} S_{ij} S_{kl} \, dx_2 = \tfrac{1}{2} c_{ijkl} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_{mn} S_{ij}^{(m)} S_{kl}^{(n)},$$

$$K \equiv \tfrac{1}{2} \int_{-h}^{h} \rho u_{j,t} u_{j,t} \, dx_2 = \tfrac{1}{2} \rho \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_{mn} u_{j,t}^{(m)} u_{j,t}^{(n)} .$$

It may be noted that

$$T_{ij}^{(n)} = \partial U / \partial S_{ij}^{(n)}; \qquad \partial S_{ij}^{(n)} / \partial S_{ji}^{(n)} = 0, \qquad i \neq j.$$

3. Truncation of series. We retain only the zero and first order components of stress and strain and we write, tentatively, from (9),

$$T_{ij}^{(0)} = 2hc_{ijkl}S_{kl}^{(0)}, \tag{10}$$

$$T_{ij}^{(1)} = \tfrac{2}{3}h^3c_{ijkl}S_{kl}^{(1)}. \tag{11}$$

The components $S_{ij}^{(0)}$ and $S_{ij}^{(1)}$ involve only the displacements $u_j^{(0)}$, $u_j^{(1)}$ and $u_j^{(2)}$ and we neglect those of higher order. Then, following Cauchy, we neglect the velocity $u_{2,t}^{(1)}$ in the kinetic energy and equations of motion and provide for the free development of the strain $S_{22}^{(0)}$ $(= u_2^{(1)})$ by setting $T_{22}^{(0)} = 0$ in (10). The condition $T_{22}^{(0)} = 0$ permits the elimination of $S_{22}^{(0)}$ from (10), with the result, again tentative,

$$T_{ij}^{(0)} = 2hg_{ijkl}S_{kl}^{(0)}, \tag{12}$$

where

$$g_{ijkl} = c_{ijkl} - c_{ij22}\,c_{22kl}/c_{2222}\,.$$

It may be observed that, because of the form of g, only five components of stress and five components of strain remain in (12).

The first order terms, $T_{ij}^{(1)}$ and $S_{ij}^{(1)}$, are treated in a similar manner except that all three velocities $u_{j,t}^{(2)}$ are neglected and free development of the three strains $S_{2j}^{(1)}$ is accommodated by setting $T_{2j}^{(1)} = 0$ in (11). Upon elimination of the $S_{2j}^{(1)}$ from (11), we arrive at

$$T_{ab}^{(1)} = \tfrac{2}{3}h^3\gamma_{abcd}S_{cd}^{(1)}; \qquad a, b, c, d = 1, 3,$$

where

$$\gamma_{abcd} = A_{abcd}/|\,s_{abcd}\,|$$

in which $|s_{abcd}|$ is the determinant

$$|\,s_{abcd}\,| = \begin{vmatrix} s_{1111} & s_{1133} & 2s_{1113} \\ s_{3311} & s_{3333} & 2s_{3313} \\ 2s_{1311} & 2s_{1333} & 4s_{1313} \end{vmatrix}$$

and A_{abcd} is the cofactor of that element of $|s_{abcd}|$ which contains s_{abcd}.

As the final step in the process of truncation, the thickness-shear strains $S_{21}^{(0)}$ and $S_{23}^{(0)}$ are replaced by $k_1S_{21}^{(0)}$ and $k_3S_{23}^{(0)}$ in the strain-energy-density, where k_1 and k_3 are correction-factors which may be used to adjust the thickness-shear frequencies to their exact values; thus compensating, in part, for the omission of terms of higher order in the series expansions. The correction-factors may be introduced conveniently by replacing g_{ijkl} with

$$g_{ijkl}^{*} \equiv k_{i+j-2}^{m}k_{k+l-2}^{n}g_{ijkl} \quad \text{(no sum)},$$

where m and n are the powers

$$m = \cos^2(ij\pi/2), \qquad n = \cos^2(kl\pi/2).$$

Thus k_{i+j-2}^{m} (or k_{k+l-2}^{n}) is equal to k_1, k_3 or 1 according as $i + j$ (or $k + l$) in g_{ijkl} is 3, 5 or neither, respectively.

4. Recapitulation. The equations remaining, after truncation of the series and adjustment of the terms retained, are [7]

*Energy-densities**

$$U = hg^*_{ijkl}S^{(0)}_{ij}S^{(0)}_{kl} + \tfrac{1}{3}h^3\gamma_{abcd}S^{(1)}_{ab}S^{(1)}_{cd}, \tag{13}$$

$$K = \rho h(u^{(0)}_{i,t}u^{(0)}_{i,t} + \tfrac{1}{3}h^2u^{(1)}_{a,t}u^{(1)}_{a,t}). \tag{14}$$

Stress-strain relations

$$T^{(0)}_{ij} = \partial U/\partial S^{(0)}_{ij} = 2hg^*_{ijkl}S^{(0)}_{kl}, \tag{15}$$

$$T^{(1)}_{ab} = \partial U/\partial S^{(1)}_{ab} = \tfrac{2}{3}h^3\gamma_{abcd}S^{(1)}_{cd}, \tag{16}$$

where $\partial S^{(0)}_{ij}/\partial S^{(0)}_{ji} = 0,\ i \neq j;\ \partial S^{(1)}_{ab}/\partial S^{(1)}_{ba} = 0,\ a \neq b$.

Stress-equations of motion

$$T^{(0)}_{ij,i} + F^{(0)}_j = 2h\rho u^{(0)}_{j,tt}, \tag{17}$$

$$T^{(1)}_{ab,a} - T^{(0)}_{2b} + F^{(1)}_b = \tfrac{2}{3}h^3\rho u^{(1)}_{b,tt}. \tag{18}$$

Strain-displacement relations

$$S^{(0)}_{ij} = \tfrac{1}{2}(u^{(0)}_{i,j} + u^{(0)}_{j,i} + \delta_{2j}u^{(1)}_i + \delta_{2i}u^{(1)}_j), \tag{19}$$

$$S^{(1)}_{ab} = \tfrac{1}{2}(u^{(1)}_{a,b} + u^{(1)}_{b,a}). \tag{20}$$

Note that, although $S^{(0)}_{22} = u^{(1)}_2$ is contained in (19), neither of the two appears in any other of the Eqs. (13)–(20) because the zero order components of strain always occur as products with g^*_{ijkl} and the latter is zero when ij or kl is 22. Hence, there are only eight components of strain and eight components of stress to be considered. The components of strain are related through the four compatibility equations

$$\begin{aligned} &S^{(0)}_{11,33} + S^{(0)}_{33,11} = 2S^{(0)}_{13,13}, \qquad S^{(1)}_{33,1} - S^{(1)}_{13,3} = S^{(0)}_{23,13} - S^{(0)}_{12,33}, \\ &S^{(1)}_{11,33} + S^{(1)}_{33,11} = 2S^{(1)}_{13,13}, \qquad S^{(1)}_{11,3} - S^{(1)}_{31,1} = S^{(0)}_{21,31} - S^{(0)}_{32,11}. \end{aligned} \tag{21}$$

Displacement-equations of motion

$$2hg^*_{ijkl}(u^{(0)}_{k,li} + \delta_{2k}u^{(1)}_{l,i}) + F^{(0)}_j = 2h\rho u^{(0)}_{j,tt}, \tag{22}$$

$$\tfrac{2}{3}h^3\gamma_{abcd}u^{(1)}_{c,da} - 2hg^*_{2bkl}(u^{(0)}_{k,l} + \delta_{2k}u^{(1)}_l) + F^{(1)}_b = \tfrac{2}{3}h^3\rho u^{(1)}_{b,tt}. \tag{23}$$

These equations are closely related to several predecessors. Thus, if the thickness-shear and flexure are eliminated by setting the transverse shear forces $T^{(0)}_{2a}$ and the couples $T^{(1)}_{ab}$ and $F^{(1)}_b$ equal to zero, the first and third of (22) reduce to the Cauchy-Voigt [1, 8] equations of low frequency extensional motion of thin plates:

$$2h\gamma_{abcd}u^{(0)}_{c,ad} + F^{(0)}_b = 2h\rho u^{(0)}_{b,tt}. \tag{24}$$

Conversely, if the extensional deformation is suppressed by setting $u^{(0)}_a = F^{(0)}_a = 0$, (22) and (23) become equations of coupled thickness-shear and flexural vibrations [2]

$$2hg^*_{a2b2}(u^{(0)}_{2,ba} + u^{(1)}_{b,a}) + F^{(0)}_2 = 2h\rho u^{(0)}_{2,tt}, \tag{25}$$

$$\tfrac{2}{3}h^3\gamma_{abcd}u^{(1)}_{c,da} - 2hg^*_{a2b2}(u^{(0)}_{2,a} + u^{(1)}_a) + F^{(1)}_b = \tfrac{2}{3}h^3\rho u^{(1)}_{b,tt}. \tag{26}$$

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*Here and in the sequel: indices i, j, k, l range over 1, 2, 3; indices a, b, c, d range over 1, 3.

Alternatively, in the case of isotropy, (22) and (23) become

$$\mu u_{b,aa}^{(0)} + (\lambda' + \mu)u_{a,ab}^{(0)} + \tfrac{1}{2}h^{-1}F_b^{(0)} = \rho u_{b,tt}^{(0)}, \tag{27}$$

$$k^2\mu(u_{2,aa}^{(0)} + u_{a,a}^{(1)}) + \tfrac{1}{2}h^{-1}F_2^{(0)} = \rho u_{2,tt}^{(0)}, \tag{28}$$

$$\mu u_{b,aa}^{(1)} + (\lambda' + \mu)u_{a,ab}^{(1)} - 3h^{-2}k^2\mu(u_{2,b}^{(0)} + u_b^{(1)}) + \tfrac{3}{2}h^{-3}F_b^{(1)} = \rho u_{b,tt}^{(1)}, \tag{29}$$

where $\lambda' = 2\mu\lambda/(\lambda + 2\mu)$ and λ and μ are Lamé's constants. Equations (27)–(29) are also the isotropic forms of (24)–(26). Equations (27) are Poisson's equations of low frequency extensional motion of thin plates [3 and 6, p. 497] or, what is the same, the equations of motion in generalized plane stress. Equations (28) and (29) are equations of flexural motion of isotropic plates with rotatory inertia and transverse shear deformation taken into account [9]. In the one-dimensional case they have the same form as the Timoshenko beam-equations [10] and in the case of equilibrium they have the same form as Reissner's plate-equations [11]. Finally, by setting the transverse shear deformation, $S_{2a}^{(0)}$, and the rotatory inertia, $\frac{2}{3}h^3\rho u_{a,tt}^{(1)}$, equal to zero, (22) and (23) may be reduced to equations equivalent to Cauchy's [1].

5. Correction factors. The values of k_1 and k_3 are found by equating the thickness-shear frequencies obtained from (23) with the corresponding ones obtained from the three-dimensional equations.

In (23), let $F_b^{(1)} = u_j^{(0)} = 0$ and $u^{(1)} = A_b^{(1)} \exp i\omega t$, where the $A_b^{(1)}$ are constants. Then (23) become

$$g_{2b2d}^{*}A_d^{(1)} = \tfrac{1}{3}\rho h^2\omega^2 A_b^{(1)}.$$

Upon equating to zero the determinant of the coefficients of the $A_b^{(1)}$, we have

$$| g_{2b2d}^{*} - g\,\delta_{bd} | = 0, \qquad g = \tfrac{1}{3}\rho h^2\omega^2 \tag{30}$$

which gives two frequencies, say ω_1 and ω_3. These are to be equated to the two lowest roots of the frequency equation

$$| c_{2i2l} - c\,\delta_{il} | = 0, \qquad c = 4\rho h^2\omega^2/\pi^2 \tag{31}$$

for the fundamental thickness-modes, as obtained from the three-dimensional equations [12]. The two equations yield a pair of equations which may be solved for k_1 and k_3 in terms of ratios of the c_{ijkl}.

As an example, consider the case of monoclinic symmetry with x_1 the digonal axis so that c_{2122} and c_{2123} are zero. Then the roots of (30) are

$$g_1 = k_1^2 g_{2121} = k_1^2 c_{2121}, \qquad \omega_1^2 = 3g_1/\rho h^2,$$

$$g_3 = k_3^2 g_{2323} = k_3^2(c_{2323} - c_{2223}^2/c_{2222}), \qquad \omega_3^2 = 3g_3/\rho h^2$$

and the two lowest roots of (31) are

$$c_1 = c_{2121}, \qquad \omega_1^2 = \pi^2 c_1/4\rho h^2,$$

$$c_3 = \tfrac{1}{2}\{c_{2222} + c_{2323} - [(c_{2222} - c_{2323})^2 + 4c_{2223}^2]^{1/2}\}, \qquad \omega_3^2 = \pi^2 c_3/4\rho h^2.$$

Equating corresponding frequencies, we find

$$k_1^2 = \pi^2/12, \qquad k_3^2 = \pi^2 c_3/12 g_{2323}. \tag{32}$$

6. Straight-crested waves in a monoclinic plate. The adequacy of Eqs. (13)–(23), for the prediction of frequencies of vibration of bounded crystal plates, may be judged by a comparison of the five-branched frequency spectrum of an infinite plate, as obtained from (22) and (23), with the first five branches of the spectrum obtained from the three-dimensional equations. This is because the modes of vibration of a bounded plate are composed essentially of the anharmonic overtones of the modes of an infinite plate, as described in Secs. 16 and 17 of Ref [13]. Because of its importance in technology, the frequency spectrum of straight-crested waves traveling along the digonal axis in the AT cut of quartz is chosen as the basis of comparison.

Quartz is a trigonal crystal (six elastic constants) and the AT cut is a plate which contains a digonal axis and whose normal makes an angle of 35°15′ with the trigonal axis [14]. When referred to rectangular axes in and normal to the plane of such a plate, the stress-strain relation has monoclinic symmetry (thirteen constants). With x_1 the digonal axis in the plane of the plate and x_2 the axis normal to the plate, the values of the thirteen constants, as computed from Bechmann's [15] values of the six principal constants, are, in units of 10^9 newtons per square meter,

$$\begin{array}{llll}
c_{1111} = 86.74 & c_{2323} = 38.61 & c_{1122} = -8.26 & c_{2233} = -7.42 \\
c_{2222} = 129.77 & c_{3131} = 68.81 & c_{1133} = 27.15 & c_{2223} = 5.70 \\
c_{3333} = 102.84 & c_{1212} = 29.01 & c_{1123} = -3.65 & c_{3323} = 9.92 \\
 & & & c_{1213} = 2.53
\end{array}$$

while

$$c_{1113} = c_{2213} = c_{3313} = c_{2313} = c_{1112} = c_{2212} = c_{3312} = c_{2312} = 0. \tag{33}$$

Then, in (22) and (23), let $F_i^{(0)} = F_b^{(1)} = 0$ and

$$u_i^{(0)} = A_i^{(0)} \exp i(\xi x_1 - \omega t), \qquad u_b^{(1)} = A_b^{(1)} \exp i(\xi x_1 - \omega t).$$

As a result, it is found that the first of (22) is coupled only with the second of (23) and the first (23) is coupled only with the second and third of (22). Thus, the general quintic, relating ω^2 and ξ^2, reduces, in this case, to a cubic and a quadratic

$$\begin{vmatrix} g_{1212}^*\xi^2 - \rho\omega^2 & g_{1312}^*\xi^2 & -ig_{1212}^*\xi \\ g_{1312}^*\xi^2 & c_{1313}\xi^2 - \rho\omega^2 & -ig_{1312}^*\xi \\ 3ih^{-2}g_{1212}^*\xi & 3h^{-2}g_{1312}\xi & \gamma_{1111}\xi^2 + 3h^{-2}g_{1212}^* - \rho\omega^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} g_{1111}\xi^2 - \rho\omega^2 & -ig_{1123}^*\xi \\ 3ih^{-2}g_{1123}^*\xi & \gamma_{1313}\xi^2 + 3h^{-2}g_{2323}^* - \rho\omega^2 \end{vmatrix} = 0.$$

The roots of these equations are plotted in Fig. 1 to dimensionless coordinates

$$\Omega = \omega/(\pi/2h)(c_{1212}/\rho)^{1/2}, \qquad \phi = 2\xi h/\pi$$

The three branches marked flexure, face-shear and thickness-shear are the roots of the cubic, while the branches marked extension and thickness-twist are the roots of the quadratic.

The five branches are to be compared with the first five branches of the analogous solution, by Ekstein [5], of the three-dimensional equations. The results of detailed

computations [16, 17] show correspondence to at least three significant figures over most of the range covered by Fig. 1. The asymptotic behaviors of the four branches which intersect at $\Omega = 0, \phi = 0$

$$\Omega = \phi^2(\pi/2)(\gamma_{1111}/3c_{1212})^{1/2} \qquad \text{(flexure)},$$

$$\Omega = \phi(\gamma_{1313}/c_{1212})^{1/2} \qquad \text{(face-shear)},$$

$$\Omega = \phi(\gamma_{1111}/c_{1212})^{1/2} \qquad \text{(extension)},$$

$$\Omega = -\phi^2(\pi/2)(\gamma_{1111}/3c_{1212})^{1/2} \qquad \text{(thickness-shear)}$$

are the same, for the approximate and exact equations, due to the method adopted in the derivation of the former. The remaining branch comes to zero frequency at

$$i\phi = (\gamma_{1111}c_3/\gamma_{1313}g_{1111})^{1/2} = 0.7456$$

in the approximate equations and $i\phi = 0.7467$ in the exact equations. In the high frequency range the important frequencies, in the usual applications, are near $\Omega = 1$ and the important branches are the thickness-shear and flexural branches. At $\Omega = 1$, the thickness-shear branch is exact, due to the choice $k_1^2 = \pi^2/12$; the flexural branch in the approximate equations has $\phi = 1.2483$ whereas the exact value is $\phi = 1.2417$.

The approximate equations should not be used for frequencies so high that branches higher than the fifth cannot be neglected. In the AT cut of quartz the sixth branch has a real minimum at a value of Ω slightly less than 2.0 [16]; however, due to the likelihood

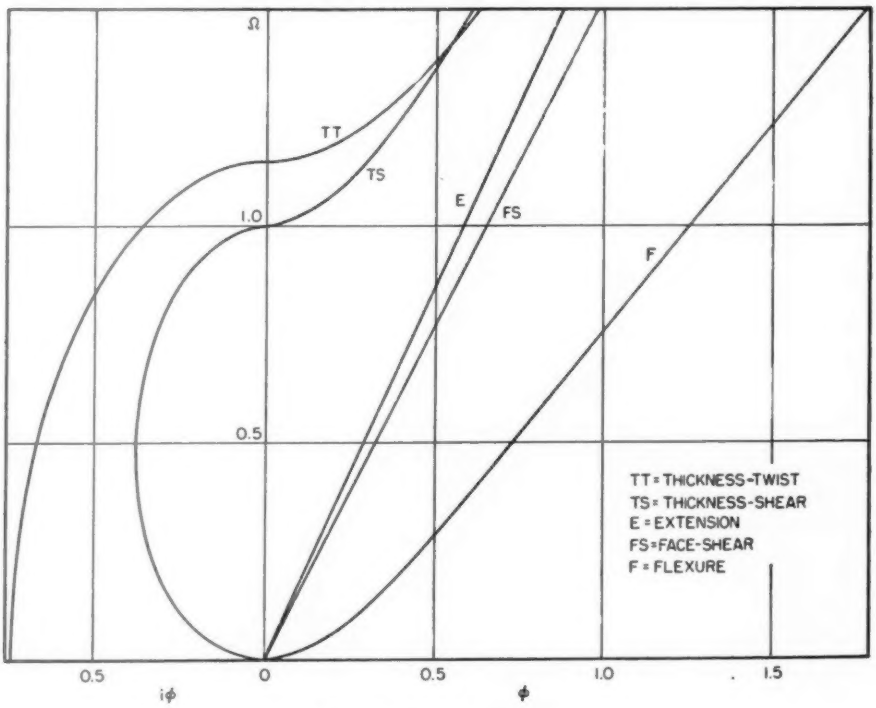

FIG. 1. Frequency spectrum.

of the contribution of an edge-mode from the complex segments of the sixth and seventh branches [17, 18], applications of the equations should be confined to $\Omega < 1.5$.

7. Uniqueness of solution. A theorem analogous to Neumann's* leads to initial and boundary conditions sufficient to assure a unique solution of equations (15)–(20).

Consider two systems of the five displacements, eight strains (omitting $S_{22}^{(0)}$) and eight stresses which satisfy the eight stress-strain equations (15)–(16), the five stress-equations of motion (17)–(18) and the eight strain-displacement equations (19)–(20). Let the differences between corresponding components of displacement, strain and stress constitute a "difference-system" of those quantities. We form the equation

$$\int_{t_0}^{t} dt \int_A [(T_{ij,i}^{(0)} + F_j^{(0)} - 2h\rho u_{j,tt}^{(0)})u_{j,t}^{(0)} + (T_{ab,a}^{(1)} - T_{2b}^{(0)} + F_b^{(1)} - \tfrac{2}{3}h^3\rho u_{b,tt}^{(1)})u_{b,t}^{(1)}]\, dA = 0 \tag{34}$$

in which all the components are those of the difference-system and t_0 is an initial time. Now

$$2h\rho \int_{t_0}^{t} dt \int_A (u_{j,tt}^{(0)}u_{j,t}^{(0)} + \tfrac{1}{3}h^2 u_{b,tt}^{(1)}u_{b,t}^{(1)})\, dA = \int_A (K - K_0)\, dA, \tag{35}$$

where K is the kinetic energy-density of the difference-system and K_0 is its value at t_0. Also

$$\begin{aligned} T_{ij,i}^{(0)}u_{j,t}^{(0)} + (T_{ab,a}^{(1)} - T_{2b}^{(0)})u_{b,t}^{(1)} &= (T_{ij}^{(0)}u_{j,t}^{(0)})_{,i} + (T_{ab}^{(1)}u_{b,t}^{(1)})_{,a} - T_{ij}^{(0)}u_{j,it}^{(0)} - T_{ab}^{(1)}u_{b,at}^{(1)} - T_{2b}^{(0)}u_{b,t}^{(1)}, \\ &= (T_{aj}^{(0)}u_{j,t}^{(0)} + T_{ab}^{(1)}u_{b,t}^{(1)})_{,a} - T_{ij}^{(0)}(u_{j,i}^{(0)} + \delta_{2i}u_j^{(1)})_{,t} - T_{ab}^{(1)}u_{b,at}^{(1)}, \\ &= (T_{aj}^{(0)}u_{j,t}^{(0)} + T_{ab}^{(1)}u_{b,t}^{(1)})_{,a} - \tfrac{1}{2}T_{ij}^{(0)}(u_{j,i}^{(0)} + u_{i,j}^{(0)} + \delta_{2i}u_j^{(1)} \\ &\quad + \delta_{2j}u_i^{(1)})_{,t} + \tfrac{1}{2}T_{ab}^{(1)}(u_{b,a}^{(1)} + u_{a,b}^{(1)})_{,t}, \\ &= (T_{aj}^{(0)}u_{j,t}^{(0)} + T_{ab}^{(1)}u_{b,t}^{(1)})_{,a} - (\partial U/\partial S_{ij}^{(0)})S_{ij,t}^{(0)} - (\partial U/\partial S_{ab}^{(1)})S_{ab,t}^{(1)}, \\ &= (T_{aj}^{(0)}u_{j,t}^{(0)} + T_{ab}^{(1)}u_{b,t}^{(1)})_{,a} - U_{,t}, \end{aligned}$$

where U is the strain-energy-density of the difference-system. Hence

$$\int_{t_0}^{t} dt \int_A [T_{ij,i}^{(0)}u_{j,t}^{(0)} + (T_{ab,a}^{(1)} - T_{2b}^{(0)})u_{b,t}^{(1)}]\, dA = \int_{t_0}^{t} dt \oint_C n_a(T_{aj}^{(0)}u_{j,t}^{(0)} + T_{ab}^{(1)}u_{b,t}^{(1)})\, ds - \int_A (U - U_0)\, dA \tag{36}$$

in which the line integral is around the boundary, C, of the plate and the n_a are the direction cosines of the outward normal in the plane of the plate. In the transformation from the surface integral to the line integral, it is assumed that the compatibility equations (21) are satisfied. Finally, using (35) and (36) in (34),

$$\int_A (U + K)\, dA = \int_A (U_0 + K_0)\, dA + \int_{t_0}^{t} dt \int_A (F_j^{(0)}u_{j,t}^{(0)} + F_b^{(1)}u_{b,t}^{(1)})\, dA + \int_{t_0}^{t} dt \oint_C n_a(T_{aj}^{(0)}u_{j,t}^{(0)} + T_{ab}^{(1)}u_{b,t}^{(1)})\, ds. \tag{37}$$

*Reference [6], p. 176.

Then conditions sufficient for a unique solution are established by the usual argument based on the vanishing of the right hand side of (37) and the positive definiteness of U and K. It may be seen that, in addition to the initial values of $u_j^{(0)}$, $u_a^{(1)}$, $u_{j,t}^{(0)}$ and $u_{a,t}^{(1)}$, there are five conditions to be specified at each point of the interior of the plate and five conditions at each point of the edge. In terms of orthogonal coordinates α, β, x_2, the interior conditions are one member of each of the five products

$$F_\alpha^{(0)} u_\alpha^{(0)}, \qquad F_\beta^{(0)} u_\beta^{(0)}, \qquad F_2^{(0)} u_2^{(0)}, \qquad F_\alpha^{(1)} u_\alpha^{(1)}, \qquad F_\beta^{(1)} u_\beta^{(1)}.$$

In terms of orthogonal coordinates n, s, x_2, the edge conditions are one member of each of the five products

$$T_{nn}^{(0)} u_n^{(0)}, \qquad T_{ns}^{(0)} u_s^{(0)}, \qquad T_{n2}^{(0)} u_2^{(0)}, \qquad T_{nn}^{(1)} u_n^{(1)}, \qquad T_{ns}^{(1)} u_s^{(1)}. \tag{38}$$

The first two of these products give the boundary conditions of generalized plane stress and the last three give the boundary conditions of Reissner's equations of flexure [11].

8. Orthogonal functions. A theorem analogous to Clebsch's* may also be established. Consider two solutions

$$(u_j^{(0)}, u_b^{(1)}) = (u_j^{(0)r}, u_b^{(1)r}) \exp i\omega_r t$$

$$(u_j^{(0)}, u_b^{(1)}) = (u_j^{(0)s}, u_b^{(1)s}) \exp i\omega_s t$$

of the homogeneous ($F_j^{(0)} = F_b^{(1)} = 0$) stress-equations of motion, so that the equations

$$\begin{aligned} 2\rho h\omega_r^2 u_j^{(0)r} &= -T_{ij,i}^{(0)r}, \\ \tfrac{2}{3}\rho h^3\omega_r^2 u_b^{(1)r} &= -T_{ab}^{(1)r} + T_{2b}^{(0)r}, \\ -2\rho h\omega_s^2 u_j^{(0)s} &= T_{ij,i}^{(0)s}, \\ -\tfrac{2}{3}\rho h^3\omega_s^2 u_b^{(1)s} &= T_{ab}^{(1)s} - T_{2b}^{(0)s} \end{aligned}$$

are satisfied. Multiplying these equations by $u_j^{(0)s}$, $u_b^{(1)s}$, $u_j^{(0)r}$, $u_b^{(1)r}$, respectively, adding and integrating over the area of the plate, we obtain, on the left hand side,

$$2h\rho(\omega_r^2 - \omega_s^2) \int_A (u_j^{(0)r} u_j^{(0)s} + \tfrac{1}{3}h^2 u_b^{(1)r} u_b^{(1)s})\, dA$$

and, on the right hand side,

$$\int_A (T_{ij,i}^{(0)s} u_j^{(0)r} - T_{ij,i}^{(0)r} u_j^{(0)s} + T_{ab,a}^{(1)s} u_b^{(1)r} - T_{ab,a}^{(1)r} u_b^{(1)s} + T_{2b}^{(0)r} u_b^{(1)s} - T_{2b}^{(0)s} u_b^{(1)r})\, dA.$$

The latter, by a process similar to that employed in the preceding section, may be transformed to

$$2h \int_A [g_{ijkl}^*(S_{ij}^{(0)s} S_{kl}^{(0)r} - S_{ij}^{(0)r} S_{kl}^{(0)s}) + \tfrac{1}{3}h^2 \gamma_{abcd}(S_{ab}^{(1)s} S_{cd}^{(1)r} - S_{ab}^{(1)r} S_{cd}^{(1)s})]\, dA$$

$$+ \oint_C n_a (T_{aj}^{(0)s} u_j^{(0)r} - T_{aj}^{(0)r} u_j^{(0)s} + T_{ab}^{(1)s} u_b^{(1)r} - T_{ab}^{(1)r} u_b^{(1)s})\, ds.$$

The integrand in the surface integral vanishes identically. The integrand in the line

*Reference [6], p. 180.

integral vanishes for any one of the thirty-two sets of edge-conditions obtained by equating to zero one member of each of the five products in (38), or for any of the associated conditions of elastic support. Then, if $w_r \neq w_s$

$$\int_A (u_j^{(0)r} u_j^{(0)s} + \tfrac{1}{3}h^2 u_b^{(1)r} u_b^{(1)s})\, dA = 0.$$

Some observations may be made regarding normal modes of rectangular plates. In the general triclinic case there are no solutions of the displacement-equations of motion, (22)–(23), with each of the five components of displacement a single product-function. However, such solutions are possible for monoclinic and higher symmetries.

Thus, in the monoclinic case, where (33) hold, (22) and (23) are satisfied by the four sets of displacements (and no others)

$$\begin{aligned}
u_1^{(0)} &= A_1^{(0)} \cos(\xi x_1 + p\pi/2) \sin(\zeta x_3 + q\pi/2) \exp i\omega t, \\
u_2^{(0)} &= A_2^{(0)} \sin(\xi x_1 + p\pi/2) \cos(\zeta x_3 + q\pi/2) \exp i\omega t, \\
u_3^{(0)} &= A_3^{(0)} \sin(\xi x_1 + p\pi/2) \cos(\zeta x_3 + q\pi/2) \exp i\omega t, \\
u_1^{(1)} &= A_1^{(1)} \cos(\xi x_1 + p\pi/2) \cos(\zeta x_3 + q\pi/2) \exp i\omega t, \\
u_3^{(1)} &= A_3^{(1)} \sin(\xi x_1 + p\pi/2) \sin(\zeta x_3 + q\pi/2) \exp i\omega t, \\
& p = 0, 1; \qquad q = 0, 1
\end{aligned} \tag{39}$$

subject to a quintic equation relating ω^2, ξ^2 and ζ^2. For a plate $x_1 = \pm h_1$, $x_3 = \pm h_3$, normal modes, in which each displacement is a single product function, are possible if and only if, on $x_1 = \pm h_1$,

$$\begin{aligned}
u_2^{(0)} &= u_3^{(0)} = u_3^{(1)} = T_{11}^{(0)} = T_{11}^{(1)} = 0 \\
\text{or} \quad T_{12}^{(0)} &= T_{13}^{(0)} = T_{13}^{(1)} = u_1^{(0)} = u_1^{(1)} = 0
\end{aligned} \tag{40}$$

and, on $x_3 = \pm h_3$,

$$\begin{aligned}
u_2^{(0)} &= u_3^{(0)} = u_1^{(1)} = T_{13}^{(0)} = T_{33}^{(1)} = 0 \\
\text{or} \quad T_{23}^{(0)} &= T_{33}^{(0)} = T_{13}^{(1)} = u_1^{(0)} = u_3^{(1)} \quad 0,
\end{aligned} \tag{41}$$

whence

$$\xi = m\pi/2h_1, \qquad \zeta = n\pi/2h_3; \qquad m, n = 0, 1, 2 \cdots \tag{42}$$

and the frequencies are given by the five roots of the quintic for each m and n.

The first set of (40) and the first set of (41) are the conditions analogous to simply supported edges in the elementary theory of flexure; but note that, in the present case, (41) are not obtained from (40) by interchange of indices 1 and 3. As in the elementary theory, solutions in closed form are not possible, in general, if all four edges are free; but closed solutions may be obtained with one pair of parallel edges free and the other pair under conditions (40) or (41). Then each of the displacements in (39) is given by a sum of five functions and one of the sets of roots (42) is replaced by the roots of a 5×5 transcendental, determinantal equation. There is one exceptional case of a solution in closed form, with all four edges free [19], analogous to the Lamé modes of an isotropic plate.

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BOOK REVIEWS

(Continued from p. 50)

The author does not discuss these matters at all. This is unfortunate, since there is little value to steady-state analysis unless one shows that the dynamic process asymptotically approaches the steady-state process as the length of the processes increases. Furthermore, it is essential to indicate the rate of approach.

The author sets himself the task of determining steady-state policies under the assumption of their existence. Granted the existence of a "steady-state," the function $f_i(n)$ has the asymptotic form $nc + b_i + o(1)$ as $n \to \infty$ where c is independent of i. The recurrence relations then yield a system of equations for c and the b_i.

This system can be studied by means of linear programming as a number of authors have realized; cf. A. S. Manne, "Linear programming and sequential decisions," *Management Science*, vol. 6, 1960, pp. 259–268.

Howard uses a different technique based upon the method of successive approximations, in this case an approximation in policy space. It is a very effective technique as the author shows by means of a number of interesting examples drawn from questions of the routing of taxicabs, the auto replacement problem and the managing of a baseball team.

The book is well written and attractively printed. It is heartily recommended for anyone interested in the fields of operations research, mathematical economics, or in the mathematical theory of Markov processes.

Richard Bellman

Handbook of supersonic aerodynamics—shock tubes. By I. I. Glass and J. Gordon Hall. Section 18. Supt. of Documents, Washington 25, D. C. xxxviii + 604 pp. $3.75.

An extensive handbook of shock tube technique. The first part, on the theory and performance of simple shock tubes, is by I. I. Glass. The second part, on the production of strong shocks and the application, design and instrumentation of shock tubes, is by J. G. Hall.

R. E. Meyer

The theory of thin elastic shells. Edited by W. Koiter. Proceedings of I. U. T. A. M. Symposium, Delft, 1959. North-Holland Publishing Company, Amsterdam, 1960. ix + 496 pp. $9.00.

The volume contains the papers presented at the Symposium on the Theory of Thin Elastic Shells sponsored by the International Union of Theoretical and Applied Mechanics (I.U.T.A.M.) and held at the Technological University of Delft in August 1959. All papers at the Symposium were delivered by invitation only and attention was confined to two particular aspects of shell theory, viz. nonlinear theory and problems lacking axial symmetry. An Introduction to the Symposium by Prof. C. B. Biezeno is followed by the contributions of the following authors: W. T. Koiter, W. Zerna, D. G. Ashwell, B. Budiansky, H. Ebner, E. I. Grigolyuk, J. H. Haywood, G. Czerwenka, W. Schnell, M. Kuranishi and J. Niisawa, A. L. Bouna, Y. Tsuboi and K. Akino, A. van der Neut, I. N. Vekua, M. W. Johnson, P. M. Naghdi, W. A. Nash and J. R. Modeer, B. R. Seth, P. Seide, N. J. Hoff and J. Singer, J. W. Choen, E. Reissner, H. Neuber.

R. T. Shield

SOLUTION OF A DIFFERENCE EQUATION PERTINENT TO LINEAR, PARAMETRIC ELECTRIC NETWORKS*

By B. J. LEON (*Hughes Research Laboratories Malibu, California*)

This paper presents an algorithm for finding the fundamental system of solutions of a class of linear difference equations relative to a theorem of Perron [1]. This particular fundamental system is important in the analysis of physical systems with periodic parameters such as linear parametric networks [2] and parametric amplifiers. These difference equations arise as the Laplace transforms of the linear differential equations with periodic coefficients that characterize parametric networks. These transforms must be analytic in a half plane. In Ref. [2] it is shown that the method of variation of parameters yields such an analytic transform if the solutions to the homogeneous equation are Perron's fundamental system. These solutions to the homogeneous equations need not be analytic or even continuous.

The homogeneous linear difference equation under consideration is

$$0 = U(\omega + n) + a_{n-1}(\omega)U(\omega + n - 1) + \cdots + a_0(\omega)U(\omega) \tag{1}$$

subject to the conditions

$$\left.\begin{aligned} &\text{a)} \quad \lim_{\substack{m\to\infty \\ \text{integer}}} a_i(\omega + m) = A_i\,, \quad \text{a constant,} \\ &\text{b)} \quad A_0 \neq 0, \quad \text{and} \quad a_0(\omega + m) \neq 0 \quad \text{for} \quad m = 0, 1, 2, \cdots, \\ &\text{c)} \quad u^n + A_{n-1}u^{n-1} + \cdots + A_0 = (u - u_1)(u - u_2) \cdots (u - u_n) \\ &\text{d)} \quad | u_i | \neq | u_j | \quad \text{for} \quad i \neq j. \end{aligned}\right\} \tag{2}$$

In the difference equation literature—Milne-Thomson [3] is the most complete reference for methods of solution—the emphasis is on finding analytic solutions to the homogeneous equation. The complexity of these methods increases markedly with both the order of the difference equation and the complexity of the coefficients of the equation. With these methods, only the simplest of parametric amplifier problems can be put on an IBM 709 computer. If a few second-order effects are included in the amplifier circuit model, the machine overflows.

The algorithm presented in this paper is intended for execution on a digital computer. When the problem is set up on a digital computer, the coefficients in Eq. (1) are subroutines, which are brought into the main program as numbers. The number of operations in the main program depends on the order of the difference equation and on the asymptotic behavior of the coefficients. The complexity of the coefficients for small ω does not affect the complexity of the main program.

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The basis for our algorithm is the property of the solutions to the difference equation given by Perron's theorem. That theorem states that for Eq. (1) subject to the conditions (2) there exists a fundamental system of solutions, $U_1, U_2, \cdots, U_n$, such that

$$\lim_{\substack{m\to\infty \\ \text{integer}}} \frac{U_i(\omega + m + 1)}{U_i(\omega + m)} = u_i \,. \tag{3}$$

We recall that the definition of a fundamental system of solutions for difference equations is similar to the definition for differential equations except that the arbitrary constants in the differential equation case are replaced by arbitrary periodic functions of period one in Re $[\omega]$.

The algorithm has five basic computational steps that are repeated over and over until all the desired solutions are found. Let us first describe the steps in words and then present them formally with justification. The steps are

(I) There is a straightforward procedure for constructing solutions to Eq. (1) [4]. Although the solutions so constructed do not have the asymptotic property (3), we may formally write each solution as a linear combination of the members of the desired fundamental system.

(II) From Eq. (3) we see that the asymptotic properties of the desired U_i are all different. Thus if we examine solutions generated in step (I) for large values of the argument we can ascertain their relative dependence on the desired U_i which has the fastest rate of growth. Call this largest solution U_1.

(III) Starting from n-linearly independent solutions constructed in step (I) with known U_1 dependence from step (II) we can proceed by systematic elimination to find $(n - 1)$ solutions which depend only on $U_2, U_3, \cdots, U_n$.

Since all the $| u_i |$ are distinct, steps (II) and (III) can be repeated eliminating the U_2 dependence, then U_3, etc., until there remains only one solution which has the asymptotic property of U_n. (We choose the subscripts so that $| u_1 | > | u_2 | > \cdots > | u_n |$.)

(IV) When a solution to an nth-order linear difference equation is known, that solution can be used in conjunction with the equation to derive a new linear difference equation of $(n - 1)$th order [5]. We shall show that if the solution U_n, found in (III), is used to derive an equation of order $(n - 1)$ then this equation has a fundamental system of solutions $U_{i,1}$ with

$$\lim_{m\to\infty} \frac{U_{i,1}(\omega + m + 1)}{U_{i,1}(\omega + m)} = \frac{u_i}{u_n}, \qquad i = 1, 2, \cdots, (n - 1).$$

The $U_{i,1}$, like the U_i, have distinct asymptotic properties. Thus we can proceed by steps (I), (II), and (III) to find a solution $U_{n-1,1}$. This solution can then be used to find an equation of order $(n - 2)$ and so on.

(V) Finally we get the desired U_i from the solution $U_{i,1}$ by

$$U_i(\omega) = U_n(\omega)\; S_c^{\omega}\; [U_{i,1}(X)]\; \Delta X,$$

where $S_c^{\omega}[\quad]\,\Delta X$ is Norlund's summing operator [6]. The various steps of the algorithm are repeated until all the desired U_i are found.

Formally the algorithm may be written as follows:

(I) Let

$$U^i(\omega) = \begin{cases} \omega^{i-1} & \text{for } 0 \le \operatorname{Re}[\omega] < n \\ -[a_{n-1}(\omega - n)U^i(\omega - 1) + a_{n-2}(\omega - n)U^i(\omega - 2) + \cdots & \\ \qquad + a_0(\omega - n)U^i(\omega - n)] & \text{recursively first} \\ & \text{for } n \le \operatorname{Re}[\omega] < (n+1) \\ & \text{then} \\ & (n+1) \le \operatorname{Re}[\omega] < (n+2) \\ & \text{etc. for all} \\ & \operatorname{Re}[\omega] \ge n \\ -\dfrac{1}{a_0(\omega)}[U^i(\omega + n) + a_{n-1}(\omega)U^i(\omega + n - 1) + \cdots & \\ \qquad + a_1(\omega)U^i(\omega + 1)] & \text{recursively first} \\ & \text{for } 0 > \operatorname{Re}[\omega] \ge -1 \\ & \text{then } -1 > \operatorname{Re}[\omega] \ge -2 \\ & \text{etc. for all } \operatorname{Re}[\omega] < 0. \end{cases}$$

Justification.

A) The U^i are well defined solutions by condition (2b) and Ref. [5].

B) The U^i form a fundamental system of solutions because

a) for $0 \le \operatorname{Re}[\omega] < 1$, Casorati's determinant,

$$D(\omega) = \begin{vmatrix} 1 & \omega & & \omega^{n-1} \\ 1 & \omega + 1 & & (\omega+1)^{n-1} \\ \vdots & & & \vdots \\ 1 & \omega + n - 1 & \cdots & (\omega + n - 1)^{n-1} \end{vmatrix}$$

is a Vandermond determinant [7], which is easily seen to be non zero.

b) By Heymann's theorem [8]

$$D(\omega + 1) = (-1)^n a_0(\omega)\, D(\omega).$$

c) By Condition (2b) above $D(\omega) \neq 0$ for all ω.

d) By Casorati's theorem[9]

$D(\omega) \neq 0 \Rightarrow$ the U^i's form a fundamental system.

C) The U^i's can be written formally as

$$\begin{aligned} U^1 &= p_1^1 U_1 + p_2^1 U_2 + \cdots + p_n^1 U_n\,, \\ U^2 &= p^2[p_1^1 U_1 + p_2^2 U_2 + \cdots + p_n^2 U_n], \\ U^3 &= p^3[p_1^1 U_1 + p_2^3 U_2 + \cdots + p_n^3 U_n], \\ &\vdots \\ U^n &= p^n[p_1^1 U_1 + p_2^n U_2 + \cdots + p_n^n U_n], \end{aligned}$$

where the p^i's and p_j^i's are periodic in Re $[\omega]$ with period one. The u_i's are ordered so that

$$|u_1| > |u_2| > \cdots > |u_n|.$$

(II) If $U^1(\omega + m) \neq 0$ let

$$p^i(\omega, m) = \frac{U^i(\omega + m)}{U^1(\omega + m)}.$$

Justification.
If $p_1^1(\omega) \neq 0$ then

$$\lim_{m\to\infty} p^i(\omega, m) = p^i(\omega).$$

To show this we note

a) By (3) for every $\epsilon > 0 \ \exists \ M_i(\omega)$ and a real, positive $K_i(\omega) \ni$ for $m < M_i(\omega)$

$$K_i(\omega)(|u_i| - \epsilon)^{m-M_i(\omega)} < |U_i(\omega + m)| < K_i(\omega)(|u_i| + \epsilon)^{m-M_i(\omega)}.$$

b) For

$$\epsilon < \frac{|u_1| - |u_2|}{2}$$

and $m > \max M_i = M_0$ we have

$$\left|\frac{U_i(\omega + n)}{U_1(\omega + m)}\right| < \frac{K_i(\omega)}{K_1(\omega)} \frac{(|u_i| + \epsilon)^{M_0 - M_i}}{(|u_1| - \epsilon)^{M_0 - M_1}} \left[\frac{|u_i| + \epsilon}{|u_1| - \epsilon}\right]^{m-M_0}.$$

So for $j \neq 1$ and $K_1(\omega) \neq 0$ we have

$$\lim_{m\to\infty} \frac{U_i(\omega + m)}{U_1(\omega + m)} = 0.$$

c) The rest is straightforward.

(III) Let

$$U^{i,1} = \frac{U^i}{p^i} - U^1 \quad \text{for} \quad i = 2, 3, \cdots, n.$$

Justification.
The $U^{i,1}$ can be written formally as

$$\begin{aligned}
U^{2,1} &= p_2^{2,1}U_2 + p_3^{2,1}U_3 + \cdots + p_n^{2,1}U_n\,, \\
U^{3,1} &= p^{3,1}[p_2^{2,1}U_2 + p_2^{3,1}U_3 + \cdots + p_n^{3,1}U_n], \\
&\vdots \\
U^{n,1} &= p^{n,1}[p_2^{2,1}U_2 + p_3^{n,1}U_3 + \cdots + p_n^{n,1}U_n],
\end{aligned}$$

where the $p^{i,1}$'s and $p_j^{i,1}$'s are rational combinations of the p^i's and p_j^i's.

This form is the same as I(c), so steps (II) and (III) can be repeated until there remains only

$$U^{n,n-1} = p_n^{n,n-1}U_n\,.$$

This solution is a member of Perron's fundamental system relative to the root u_n. Henceforth we suppress the superscripts and refer to this solution as U_n.

(IV) Consider the equation

$$U_{,1}(\omega + n - 1) + a_{n-2,1}(\omega)U_{,1}(\omega + n - 2) + \cdots + a_{0,1}(\omega)U_{,1}(\omega) = 0 \tag{4}$$

with

$$-a_{s,1}(\omega) = a_0(\omega)U_n(\omega) + \cdots + a_s(\omega)U_n(\omega + s).$$

Justification.

A) Equation (4) has a fundamental system of solutions [5]

$$U_{i,1} = \Delta \frac{U_i}{U_n}, \qquad i = 1, 2, \cdots, (n-1),$$

where

$$\Delta f(X) = f(X+1) - f(X).$$

B) $$\lim_{m\to\infty} \frac{U_{i,1}(\omega + n + 1)}{U_{i,1}(\omega + m)} = \frac{u_i}{u_n}.$$

To show this we write

$$\frac{U_{i,1}(\omega + m + 1)}{U_{i,1}(\omega + m)} = \left[\frac{U_i(\omega + m + 2)}{U_n(\omega + m + 2)} - \frac{U_i(\omega + m + 1)}{U_n(\omega + m + 1)}\right] \cdot \left[\frac{U_i(\omega + m + 1)}{U_n(\omega + m + 1)} - \frac{U_i(\omega + m)}{U_n(\omega + m)}\right]^{-1},$$

$$= \left\{\left[\frac{U_i(\omega + m + 2)}{U_i(\omega + m)}\right] \Big/ \left[\frac{U_i(\omega + m + 1)}{U_i(\omega + m)}\right] - \left[\frac{U_n(\omega + m + 2)}{U_n(\omega + m)}\right] \Big/ \left[\frac{U_n(\omega + m + 1)}{U_n(\omega + m)}\right]\right\} \cdot \left\{\left(\left[\frac{U_i(\omega + m + 1)}{U_i(\omega + m)}\right] \Big/ \left[\frac{U_n(\omega + m + 1)}{U_n(\omega + m)}\right]\right) - 1\right\}^{-1}$$

Thus

$$\lim_{m\to\infty} \frac{U_{i,1}(\omega + m + 1)}{U_{i,1}(\omega + m)} = \frac{(u_i^2/u_n^2) - (u_i/u_n)}{(u_i/u_n) - 1} = \frac{u_i}{u_n}.$$

C) The construction procedure (I) gives a fundamental system of solutions to Eq. (4). By Heymann's theorem the only zeros of $D(\omega)$ are congruent to the zeros of $a_{0,1}$, but these are considered singular points of the equation. Thus the hypotheses of Casorati's theorem are satisfied.

D) Repeated application of (II) and (III) leads to a solution with the asymptotic property of $U_{(n-1),1}$. This solution, along with Eq. (4), can be used to construct an equation of order $(n - 2)$ with the same properties as Eq. (4).

(V) Let

$$U_i = U_n\, S_c^{\omega}\, [U_{i,1}(X)]\, \Delta X,$$

where $\int_c^{\omega} [\quad] \Delta X$ is Norlund's summing operator [6].

Justification.

These U_i are the solutions U_i to Eq. (1) that we desire because $\int_c^\omega$ is the inverse of Δ (step (IV)A) to within a periodic function. Thus the problem is solved.

Discussion. In the various steps of the algorithm divisions are required. For these operations we must be sure that the divisor is non zero. First we note that if the coefficient of Eq. (1) are analytic along a line parallel to the real ω axis except for isolated singularities, then the solutions U^i generated in step (I) will have the same property along that line except at the integer values of Re $[\omega]$. Therefore the zeros of the various divisors will be discrete, and the algorithm can fail because of attempted division by zero only at a discrete set of points.

The other place where the algorithm as stated may fail is in step (II) where we require $p_1^1(\omega) \neq 0$. It is possible that our construction of the U^i gives a U^1 which is independent of U_1. Since the U^i form a fundamental system, at least one of the U^i must depend on U_1. Therefore if we find that the $p^i(\omega, n)$ do not tend to a limit, we must renumber the U^i so that the new U^1 is one of the U^i that depends on U_1. Throughout the elimination process there is the possibility of this analogous situation occurring. However, it can always be circumvented by renumbering the U'.

In the parametric network problem there are symmetries in the original homogeneous equation. These symmetries can be used to reduce the number of operations required in the algorithm. In a subsequent paper these symmetries and the corresponding reductions will be discussed.

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—NOTES—

A NOTE ON THE MINIMAL REPRESENTATION OF TRIGGERING MATRICES*

BY

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Abstract. A triggering matrix is an array of numbers in which each row represents an actuating input of a data-processing system. In this note, upper and lower bounds are given to the minimal form of such a matrix.

In this note the term "triggering set" will refer to any group of differing numbers, each of which serves as an input to a computing or control system. The members of the set, called "trigger-words", have no numerical significance—their only function being the initiation of various processes in the system. It will be assumed that each trigger-word initiates a different process, and hence that the only requirement for satisfactory system operation is the ability to differentiate among the words of the set. Typically, a triggering set may be the digital representation of analog quantities to be fed into a recognition system.

In the following discussion, m will denote the number of trigger-words in the set, n' the number of digits in each word, and r the "set radix"—the number of values that each digit can assume. m, n' and r will be taken as finite. For convenience, a triggering set will be written as a matrix, called the "triggering matrix", in which the element common to the ith row and jth column will represent the jth digit in the ith word. Matrix (1) shown below is an example of a triggering matrix with $m = 6$, $n' = 9$ and $r = 3$.

$$\begin{array}{c c}
 & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\begin{bmatrix}
2 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 \\
2 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\
1 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 1 \\
0 & 2 & 1 & 1 & 0 & 2 & 1 & 1 & 2 \\
0 & 2 & 1 & 0 & 2 & 2 & 1 & 1 & 0
\end{bmatrix}
\end{array} \tag{1}$$

A problem which often arises in data-processing systems with inputs characterizable by triggering matrices, is that of determining the smallest number of digits necessary

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for satisfactory system operation. In terms of the triggering matrix, it is desired to find the smallest number of columns such that the rows are still distinct. Under noisy conditions it is further desired to maintain a specified minimal distance between the rows in order to increase reliability ("distance" having the usual Hamming connotation, extended to codes of radix r). Deleting the largest number of columns from a triggering matrix such that the distance between any two rows is not less than d, results in a matrix which will be called the "minimal triggering matrix consistent with d". In a previous paper [1] the bounds on this deletion were produced for the special case $r = 2$ (i.e., for binary triggering matrices). The following is the extension of the previous results to the general case.

Theorem. Let the dimension of a triggering matrix of radix r and minimal distance d_0 be $m \times n'$. Let the dimension of the corresponding minimal triggering matrix consistent with the minimal distance $d \leq d_0$ be $m \times n$. Then

$$\left[\frac{d-1}{r-1}\right] + \{\log_r m\} \leq n \leq d(m-1),$$

where $[x]$ denotes the largest integer which is not larger than x, and $\{x\}$ the smallest integer which is not smaller than x.

The lower bound equals the minimum number of columns which can accommodate radix-r rows with minimal distance d. This number can be found by generalizing relationships previously derived in [2] and [3] for the binary minimal distance codes. For that purpose, let α be a sequence of n digits of radix r

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \qquad \alpha_1 = 0, \quad \text{or} \quad 1, \cdots, \quad \text{or} \quad r-1.$$

The sequence $\alpha \ominus \beta$ will be defined by:

$$\alpha \ominus \beta = (|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|, \cdots, |\alpha_n - \beta_n|).$$

The distance between α and β, denoted $|\alpha \ominus \beta|$, is then:

$$|\alpha \ominus \beta| = \sum_{i=1}^{n} |\alpha_i - \beta_i|.$$

Now, let M be an m-word n-digit code of radix r and minimal distance d. Let α, β ($\alpha \neq \beta$) be any n-digit words of radix r which end with $n - [d - 1/r - 1]$ zeros. Then:

$$|\alpha \ominus \beta| \leq (r-1)\left[\frac{d-1}{r-1}\right] \leq d-1.$$

If γ, δ ($\gamma \neq \delta$) are any elements of M, then $\gamma \ominus \alpha$ and $\delta \ominus \beta$ cannot be identical. Consequently:

$$m \cdot r^{[d-1/r-1]} \leq r^n$$

or:

$$n \geq \left[\frac{d-1}{r-1}\right] + \log_r m.$$

Since n must be an integer, the lower bound of the theorem follows.

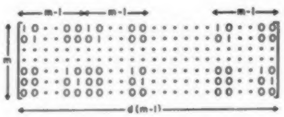
To verify the upper bound, consider an $m \times n'$ $\{\log_2 r\}$ binary matrix produced by

replacing each element in the original triggering matrix with its binary equivalent. This is done in matrix (2) for the ternary triggering matrix (1).

$$\begin{array}{cc} & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1\,0 & 0\,1 & 1\,0 & 0\,0 & 0\,1 & 0\,1 & 1\,0 & 0\,0 & 0\,1 \\ 1\,0 & 0\,1 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 1\,0 & 1\,0 & 0\,0 \\ 0\,1 & 0\,0 & 1\,0 & 1\,0 & 1\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 \\ 0\,1 & 0\,0 & 0\,0 & 1\,0 & 0\,1 & 0\,1 & 0\,1 & 1\,0 & 0\,1 \\ 0\,0 & 1\,0 & 0\,1 & 0\,1 & 0\,0 & 1\,0 & 0\,1 & 0\,1 & 1\,0 \\ 0\,0 & 1\,0 & 0\,1 & 0\,0 & 1\,0 & 1\,0 & 0\,1 & 0\,1 & 0\,0 \end{bmatrix} \end{array} \tag{2}$$

In the new matrix there are n' groups of $\{\log_2 r\}$ columns, each column group representing a single column in the original matrix. Now, from previous results (see [1]), it is known that the minimal binary matrix consistent with d contains at most $d(m-1)$ columns, and hence at most $d(m-1)$ column groups. Converting these selected column groups back to their original radix-r representation results, then, in a matrix of radix r which contains not more than $d(m-1)$ columns and whose minimal distance is at least d. The upper bound is thus verified.

Both the lower and upper bounds in the theorem are achievable with equality. The lower bound is attained, for instance, when m is a power of r, $d = 1$, and the triggering set contains all possible radix-r words of length $\log_r m$. A triggering matrix in which the upper bound is attained is shown in (3).



(3)

It can be concluded that both bounds on the minimized number of digits are independent of the original number of digits, and that the upper bound is also independent of the number of values which each digit can assume.

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ON OPTIMUM RECTANGULAR COOLING FINS*

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1. Introduction. In the past, most discussions concerning the cooling fins of a heat exchanger were confined to cases where the heat from the fin surface was eventually dissipated by the surrounding convective fluid. For this case, the governing equation is linear and the solution can be obtained without difficulty [1]. However, as space technology advances, a heat exchanger may have to be designed for an environment where the only heat transfer mechanism is by radiation. Furthermore, in any space vehicle design, the over-all weight of the vehicle is of utmost importance. It is, therefore, desirable to know the fin geometry of least weight. The essential difficulty in dealing with cooling fins when the convective transfer mechanism becomes insignificant arises from the fact that the governing equation is no longer linear. Up to the present, the solutions for this type of problem were obtained mainly by numerical method [2, 3]. It is difficult to optimize the result thus obtained.

The present paper presents a parametric solution for a rectangular cooling fin in terms of known functions, from which the optimum geometry of the fin with least weight is uniquely determined.

2. Statement of the problem. The governing differential equation of the temperature $T(x)$ in a rectangular thin fin as shown in Fig. 1 is

$$\frac{d^2T}{dx^2} - \frac{2s}{kb} T^\alpha = 0, \tag{1}$$

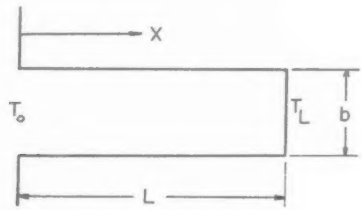

FIG. 1

where

s = heat transfer coefficient,
k = conductivity of the fin material,
b = fin width,
α = a constant, equal to $1 \rightarrow 4$ in actual application and assumed to be greater than unity in the present problem.

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The boundary conditions for a long fin are

$$T = T_0 \quad \text{at} \quad x = 0 \tag{2}$$

$$\frac{dT}{dx} = 0 \quad \text{at} \quad x = L. \tag{3}$$

The cooling rate of the fin in terms of the rate of heat conducted out of the fin base is

$$q = -kb \left.\frac{dT}{dx}\right|_{x=0}. \tag{4}$$

The problem may now be stated as: For a fixed fin weight (or area $A = bL$), determine the value of b such that q is a maximum subject to the governing equation (1) and the boundary conditions (2) and (3).

3. The temperature distribution. Though the solution of the system of Eqs. (1), (2) and (3) can not be obtained in closed form, it is possible to derive a parametric relation between T and x as shown below.

Multiplying Eq. (1) by dT/dx and integrating gives

$$\left(\frac{dT}{dx}\right)^2 - \frac{4s}{(\alpha + 1)kb} T^{\alpha+1} + C = 0. \tag{5}$$

The constant C is determined by Eq. (3) and by assuming

$$T = T_L \quad \text{at} \quad x = L. \tag{6}$$

Then,

$$C = \frac{4s}{(\alpha + 1)kb} T_L^{\alpha+1}. \tag{7}$$

Note that the boundary value problem is completely defined by Eqs. (1), (2) and (3). The additional condition, Eq. (6), is used merely as a parameter which shall be uniquely determined. From Eqs. (5) and (7)

$$\frac{dT}{dx} = -\left[\frac{4s}{(\alpha + 1)kb}\right]^{1/2} T^{(\alpha+1)/2}\left[1 - \left(\frac{T_L}{T}\right)^{\alpha+1}\right]^{1/2}. \tag{8}$$

Introducing the new variables

$$t = (T_L/T)^{\alpha+1}, \tag{9}$$

$$t_0 = (T_L/T_0)^{\alpha+1}. \tag{10}$$

Equation (8) becomes

$$t^{\beta-1}(1 - t)^{-1/2} \frac{dt}{dx} = \left[\frac{4(\alpha + 1)s}{kb}\right]^{1/2} T_L^{(\alpha-1)/2}, \tag{11}$$

where $\beta = (\alpha - 1)/2(\alpha + 1)$.

Integrating Eq. (11) gives

$$\int_t^1 \eta^{\beta-1}(1 - \eta)^{-1/2}\, d\eta = \left[\frac{4(\alpha + 1)s}{kb}\right]^{1/2} T_L^{(\alpha-1)/2} \int_x^L dx \tag{12}$$

or

$$B(\beta, \tfrac{1}{2}) - B_t(\beta, \tfrac{1}{2}) = \left[\frac{4(\alpha + 1)s}{kb}\right]^{1/2} T_L^{(\alpha-1)/2}(L - x), \tag{13}$$

where B is the complete Beta function and B_t, the incomplete Beta function which is defined for $0 \leq t \leq 1$ and $\beta > 0$ [4]. The two conditions for the existence of the Beta function will be examined with regard to Eq. (13):

(i) when $t = 0$, Eq. (13) fails to give a meaningful solution. However, for this case, either $T \to \infty$ which is physically impossible or $T_L = 0$ which will be discussed at the last section as a special case. For the present discussion, we assume $0 < t \leq 1$ in Eq. (13).

(ii) when $\beta > 0$, $\alpha > 1$. If $\alpha = 1$, the problem is for the case with the ordinary convective surface condition, for which the solution is available [1].

Under the restrictions of $0 < t \leq 1$ and $\alpha > 1$, Eq. (13) expresses the functional relation between T and x with T_L as the parameter. Furthermore, the values of T and x have a one-to-one correspondence, since both sides of Eq. (13) are single-valued functions. To determine the parameter T_L, Eq. (2) is substituted into Eq. (13).

$$B(\beta, \tfrac{1}{2}) - B_{t_0}(\beta, \tfrac{1}{2}) = \left[\frac{4(\alpha + 1)sL^2}{kb}\right]^{1/2} T_L^{(\alpha-1)/2}. \tag{14}$$

After T_L is calculated from Eq. (14) for a fixed physical condition, the rate of heat transfer, Eq. (4), becomes

$$q = \left[\frac{4skb}{(\alpha + 1)}\right]^{1/2} (T_0^{\alpha+1} - T_L^{\alpha+1})^{1/2}. \tag{15}$$

4. The optimum geometry. Equation (15) gives the rate of heat transfer in terms of the parameter T_L which has to be determined from Eq. (14). For a fixed $A = bL$, T_L will vary with b, so that q is, in general, a function of b and T_L. If Eqs. (14) and (15) are rewritten as

$$q(b, T_L) = \left(\frac{4sk}{\alpha + 1}\right)^{1/2} b^{1/2}(T_0^{\alpha+1} - T_L^{\alpha+1})^{1/2}, \tag{16}$$

$$p(b, T_L) = B(\beta, \tfrac{1}{2}) - B_{t_0}(\beta, \tfrac{1}{2}) - \left[\frac{4(\alpha + 1)sA^2}{k}\right]^{1/2} b^{-3/2}T_L^{(\alpha-1)/2} = 0, \tag{17}$$

where q and p are now to be considered as functions of b and T_L, the equivalent problem of optimizing q will be to find the extreme values of the function $q(b, T_L)$ subject to the subsidiary condition $p(b, T_L) = 0$. The stationary point thus found gives the desired optimum values of b and T_L. The solution to this problem can be obtained by means of Lagrange's multiplier m defined by [5]

$$\frac{\partial q}{\partial b} + m\frac{\partial p}{\partial b} = 0, \tag{18}$$

$$\frac{\partial q}{\partial T_L} + m\frac{\partial p}{\partial T_L} = 0. \tag{19}$$

Equations (17), (18) and (19) serve to determine the stationary values of b, T_L and the constant m. Eliminating m from Eqs. (18) and (19) gives

$$b^{1/2} = -\left[\frac{(\alpha - 1)^2 sA^2}{(\alpha + 1)kT_0^2}\right]^{1/6} \frac{[T_0^{\alpha+1} - 2(2\alpha + 1)(\alpha - 1)^{-1}T_L^{\alpha+1}]^{1/3}}{(T_0^{\alpha+1} - T_L^{\alpha+1})^{1/6}}. \tag{20}$$

Comparing Eqs. (16) and (20), it is seen that $t_0 > (\alpha - 1)/2(2\alpha + 1)$. Though Eqs. (17) and (20) are sufficient to determine the optimum value of b by eliminating T_L, it is more convenient to write Eq. (20) as

$$t_0 = \frac{1}{8(2\alpha + 1)} [4(\alpha - 1) - G + G^{1/2}\{24(\alpha + 1) + G\}^{1/2}], \tag{21}$$

where

$$G = (\alpha + 1)kb^3/(2\alpha + 1)sA^2T_0^{\alpha-1} \tag{22}$$

and the positive sign in front of $G^{1/2}$ is chosen because $t_0 > (\alpha - 1)/2(2\alpha + 1)$. Equation (17) becomes

$$B(\beta, \tfrac{1}{2}) - B_{t_0}(\beta, \tfrac{1}{2}) = 2(\alpha + 1)(2\alpha + 1)^{-1/2}G^{-1/2}t_0^{\beta}, \tag{23}$$

where t_0 is given by Eq. (21). It is seen that the transcendental Equation (23) contains only the variables α and G. For a fixed $\alpha > 1$ and under the conditions $G > 0$ and $t_0 > (\alpha - 1)/2(2\alpha + 1)$, Eq. (23) is satisfied by one, and only one, value of G which, in turn, determines uniquely the optimum geometry of the fin from Eq. (22).

It remains to be shown that the extreme value of the function $q(b, T_L)$ does exist at the stationary point thus found and that the extreme value is a true maximum. For the existence of the extreme value, it is necessary that the two partial derivatives $\partial p/\partial b$ and $\partial p/\partial T_L$ shall not both vanish at the stationary point [5]. It can easily be shown that both derivatives will vanish only when $T_L = 0$ which is excluded in the present problem. The sufficient condition for the extreme value of q to be a maximum is that $d^2q < 0$ at the stationary point. This is indeed true if $t_0 > (\alpha - 1)/2(2\alpha + 1)$.

5. Special case when $T_L = 0$. In the above discussion, the case when $T_L = 0$ is excluded. If $T_L = 0$, $C = 0$ from Eq. (7). A direct integration of Eq. (5) with boundary condition (2) gives

$$T^{-(\alpha-1)/2} - T_0^{-(\alpha-1)/2} = \left[\frac{(\alpha - 1)^2 s}{(\alpha + 1)kb}\right]^{1/2} x \tag{24}$$

$$q = \left(\frac{4skb}{\alpha + 1}\right)^{1/2} T_0^{(\alpha+1)/2}. \tag{25}$$

It is seen that no maximum value of q exists for any finite value of b. The non-existence of the extreme value is apparent since physically $T_L = 0$ requires the fin length $L \to \infty$.

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GENERALIZED THERMAL RESISTANCE

BY

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1. Introduction. The thermal resistance of a conducting body V is usually defined (see, e.g., [3]**) in terms of a harmonic function $u(x_1, x_2, x_3)$, satisfying the boundary conditions

$$u = \begin{cases} u_1, & \text{on } S_1, \\ u_0, & \text{on } S_0, \end{cases}$$

$$u_{,i}n_i = 0, \quad \text{on } S_a,$$

where $S = S_1 + S_0 + S_a$ is the boundary of V, n_i is the outward normal vector, and u_1 and u_0 are constants. We shall refer to S_1 as the inlet, S_0 as the outlet, and S_a as the adiabatic surface of the body V.

The thermal resistance R of this configuration, i.e., the body V with given inlet and outlet, is then defined by the equation

$$R(u) = (u_1 - u_0)/Q(u), \tag{1.1}$$

where

$$Q(u) = \int_{S_1} u_{,i}n_i \, dS,$$

the total transmitted heat power.

By applying Green's identity to Eq. (1.1), using the harmonicity of u in V, we obtain the alternate forms

$$R(u) = (u_1 - u_0)^2/D(u) = D(q_i)/Q^2(q_i), \tag{1.2}$$

where $q_i = u_{,i}$, $Q(q_i) = Q(u)$, and the Dirichlet integrals D are given by

$$D(u) = D(q_i) = \int_V q_i q_i \, dV = \int_V u_{,i}u_{,i} \, dV.$$

If $u_1 - u_0$ is interpreted as a potential difference and Q as total current, Eqs. (1.1) and (1.2) also give the electrical resistance of the configuration for unit electrical conductivity. The two forms given by Eq. (1.2) are simply the familiar "E^2/R", "I^2R" relations between the power dissipation D, the potential drop $E = u_1 - u_0$, and the total current, $I = Q(u)$.

Since $u_1 - u_0$ is obviously the maximum temperature difference in the body V, $R(u)$ gives the maximum temperature difference for unit transmitted heat power and thus serves as a power rating for the body, at least for these boundary conditions. Unfortu-

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**Numbers in square brackets refer to the bibliography.

nately, in contrast to the problem of electrical conduction, in practice one encounters a wide variety of boundary conditions in heat flow problems. We shall show, in fact, that the temperature rise in general is greater for given input heat power than that given by the above Dirichlet conditions, so that $R(u)$ is unsafe to use as a power rating, unless, of course, it is certain that the above boundary conditions are satisfied.

Very often in practical problems the exact boundary conditions are unknown. In the following, we consider two other boundary value problems, which we regard as typical, namely the Neumann problem and the Robin-Neumann problem, both with piecewise constant boundary values. We extend the definition of thermal resistance to these cases in a natural and expeditious way and obtain upper and lower bounds, which include the resistance in all three problems between them and thus serve as universal bounds, at least for the class of problems considered.

2. The Neumann problem. The temperature v, again harmonic in V, now satisfies the boundary conditions

$$v_{,i}n_i = \begin{cases} Q/S_1\,, & \text{on} \quad S_1\,, \\ 0, & \text{on} \quad S_a\,, \\ -Q/S_0\,, & \text{on} \quad S_0\,, \end{cases}$$

where Q is the total input heat power. In this case the thermal resistance is not usually defined. We may extend the definition, however, by using the analogy with electrical conduction and requiring that the resistance $R(v)$ again satisfy the usual "I^2R" relation between the power dissipation D and the total current Q, namely

$$D(v) = Q^2R(v).$$

If we apply Green's identity, we obtain

$$R(v) = (\langle v\rangle_1 - \langle v\rangle_0)/Q, \tag{2.1}$$

where the average inlet and outlet temperatures $\langle v\rangle_1$ and $\langle v\rangle_0$ are given by

$$\langle v\rangle_1 = S_1^{-1}\int_{S_1} v\,dS, \qquad \langle v\rangle_0 = S_0^{-1}\int_{S_0} v\,dS.$$

As before, we have the alternate forms

$$R(v) = (\langle v\rangle_1 - \langle v\rangle_0)^2/D(v) = D(q_i)/Q^2, \tag{2.2}$$

where now $q_i = v_{,i}$.

If we compare Eqs. (1.1) and (2.1), we see that we can include both in a single definition, if we define the resistance in general as the ratio of the average inlet-outlet temperature difference to the total input heat power. This is the definition which we shall adopt. It is particularly apt because it admits immediately of a simple physical interpretation and because, at least in the two cases considered so far, it may be expressed in terms of the Dirichlet integral which is easily bounded.

3. The Robin-Neumann problem. We now consider a heat flow problem which we regard as canonical and from which the two preceding problems may be derived as special cases. We again seek a harmonic function w, now satisfying the boundary conditions

$$w_{,i}n_i = \begin{cases} h_1(w_1 - w), & \text{on } S_1 , \\ 0, & \text{on } S_a , \\ h_0(w_0 - w), & \text{on } S_0 , \end{cases}$$

where w_1 and w_0 are constant inlet and outlet ambient temperatures and h_1 and h_0 are positive, constant heat transfer coefficients.

These boundary conditions approximate a wide variety of modes of surface heat transfer, including radiation, convection, and contact resistance. For large h the boundary conditions approach the Dirichlet conditions satisfied by u and for small h (with some reservations as to the character of the limiting process) the Neumann conditions satisfied by v. We therefore expect that in this case the resistance will have a value intermediate between $R(u)$ and $R(v)$, namely

$$R(u) \le R(w) \le R(v), \tag{3.1}$$

where

$$R(w) = (\langle w \rangle_1 - \langle w \rangle_0)/Q(w),$$

according to our general definition of resistance. In the next section we obtain upper and lower bounds on $R(u)$ and $R(v)$ which we then use to derive the inequality (3.1).

4. Upper and lower bounds on resistance. The problem of bounding the resistances $R(u)$ and $R(v)$ is equivalent to bounding the corresponding Dirichlet integrals. Such bounds may be obtained in a variety of ways (see, e.g., [2], [3], [4]). For our purposes, the most direct approach is the use of Schwarz's inequality, as suggested by Diaz and Weinstein [1]. We first state the results and the conditions on the bounding functions and then give a brief derivation of one of them.

We have

$$(u_1 - u_0)^2/D(f) \le R(u) \le D(p_i)/Q^2(p_i), \tag{4.1}$$

$$(\langle f \rangle_1 - \langle f \rangle_0)^2/D(f) \le R(v) \le D(p_i)/Q^2, \tag{4.2}$$

where, as before,

$$D(f) = \int_V f_{,i} f_{,i} \, dV, \qquad D(p_i) = \int_V p_i p_i \, dV,$$

$$Q(p_i) = \int_{S_1} p_i n_i \, dS,$$

$$\langle f \rangle_1 = S_1^{-1} \int_{S_1} f \, dS, \qquad \langle f \rangle_0 = S_0^{-1} \int_{S_0} f \, dS.$$

In these inequalities f is any non-constant scalar function, continuous on $V + S$, piecewise continuously differentiable in V, and satisfying any boundary conditions of Dirichlet type given on S. Thus in (4.1)

$$f = \begin{cases} u_1 , & \text{on } S_1 , \\ u_0 , & \text{on } S_0 , \end{cases}$$

while in (4.2) it is unrestricted. The vector p_i is any non-zero, solenoidal vector function, having a continuous normal component across any surface in V, piecewise continuous derivatives in V, and such that its normal component satisfies any given Neumann

conditions on S. Thus in both (4.1) and (4.2)

$$p_{i,i} = 0, \quad \text{in} \quad V,$$
$$p_i n_i = 0, \quad \text{on} \quad S_e,$$

while, in addition in (5.2),

$$p_i n_i = \begin{cases} Q/S_1, & \text{on} \quad S_1 \\ -Q/S_0, & \text{on} \quad S_0 \end{cases}.$$

As a sample, let us derive the upper bound on $R(u)$. For two arbitrary vector functions, p_i and q_i, Schwarz's inequality takes the form

$$[D(p_i, q_i)]^2 \leq D(p_i)\, D(q_i), \tag{4.3}$$

where, using obvious notation,

$$D(p_i, q_i) = \int_V p_i q_i \, dV.$$

If we set $q_i = u_{,i}$, this becomes

$$[D(p_i, u_{,i})]^2 \leq D(p_i)\, D(u).$$

Green's identity gives

$$D(p_i, u_{,i}) = \int_S u p_i n_i \, dS = (u_1 - u_0) Q(p_i),$$

using the boundary conditions on p_i and u. Thus

$$(u_1 - u_0)^2 / D(u) \leq D(p_i)/Q^2(p_i),$$

which is the required inequality. The remaining bounds follow in so similar a fashion that there is no need to reproduce their derivation here.

Now note that we may set $p_i = w_{,i}$ in the upper bound on $R(u)$ and $f = w$ in the lower bound on $R(v)$, giving

$$R(u) \leq R(w)\; D(w)/Q(w)(\langle w\rangle_1 - \langle w\rangle_0),$$
$$R(v) \geq R(w) Q(w)(\langle w\rangle_1 - \langle w\rangle_0)/D(w).$$

We now show that

$$D(w) \leq Q(w)(\langle w\rangle_1 - \langle w\rangle_0),$$

from which the inequality

$$R(u) \leq R(w) \leq R(v) \tag{4.4}$$

follows.

Green's identity gives

$$Q(w) = \int_{S_1} w_{,i} n_i \, dS = h_1 S_1 (w_1 - \langle w\rangle_1)$$
$$= -\int_{S_0} w_{,i} n_i \, dS = -h_0 S_0 (w_0 - \langle w\rangle_0),$$

and

$$D(w) = h_1 S_1 w_1 \langle w \rangle_1 + h_0 S_0 w_0 \langle w \rangle_0 - h_1 \int_{S_1} w^2 \, dS - h_0 \int_{S_0} w^2 \, dS.$$

Schwarz's inequality gives

$$S_1 \int_{S_1} w^2 \, dS \geq \left[\int_{S_1} w \, dS \right]^2$$

or

$$h_1 \int_{S_1} w^2 \, dS \geq h_1 S_1 \langle w \rangle_1^2$$

and similarly for the integral over S_0, so that

$$D(w) \leq h_1 S_1 \langle w \rangle_1 (w_1 - \langle w \rangle_1) + h_0 S_0 \langle w \rangle_0 (w_0 - \langle w \rangle_0) = Q(w)(\langle w \rangle_1 - \langle w \rangle_0),$$

which is the desired inequality.

5. Conclusion. We have now shown that the thermal resistance, i.e., the average inlet-outlet temperature difference for unit input heat power, in the case of the Robin-Neumann problem is bounded below by the resistance $R(u)$ for constant inlet and outlet temperatures and above by the resistance $R(v)$ for constant inlet and outlet heat flux. We have also shown how upper and lower bounds on $R(u)$ and $R(v)$ may be obtained. Thus we have a method for estimating the temperature rise which is relatively independent of the specific nature of the boundary conditions. It is felt that this method of obtaining broad bounds on temperature rise in many cases is much more practical than, for example, the calculation of an exact solution. It is particularly suited to the kind of problem encountered, for example, in preliminary design work, where one knows little about the specific nature of the heat flow and where one may want to evaluate the effect of changes in many design parameters.

From the mathematical point of view, the foregoing results are quite trivial. It is interesting to note, however, that they provide another illustration of the complementary nature of the Dirichlet and Neumann conditions. According to the Dirichlet principle, among all functions having given constant values on inlet and outlet, the harmonic function u makes the Dirichlet integral $D(u)$ smallest. If we relax this condition and require only that the admissible functions have a given difference of average values between inlet and outlet, the Dirichlet integral is minimized by the harmonic function v. The first result gives the lower bound on $R(u)$, the second the lower bound on $R(v)$. The upper bounds may be similarly described, using the Thomson principle concerning solenoidal vectors.

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